The Nakano–Nishijima–Gell-Mann Formula from Discrete Galois Fields

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Abstract: The well known Nakano–Nishijima–Gell-Mann (NNG) formula relates certain quantum numbers of elementary particles to their charge number. This equation, which phenomenologically introduces the quantum numbers I_z (isospin), S (strangeness), etc., is constructed using group theory with real numbers $\mathbb{R}$. But, using a discrete Galois field $\mathbb{F}_p$ instead of $\mathbb{R}$ and assuring the fundamental invariance laws such as unitarity, Lorentz invariance, and gauge invariance, we derive the NNG formula deductively from Meson (two quarks) and Baryon (three quarks) representations in a unified way. Moreover, we show that quark confinement ascribes to the inevitable fractionality caused by coprimeness between half-integer (1/2) of isospin and number of composite particles (e.g., three).

1. Introduction

The standard model of particle physics has been established based on physics with real numbers ($\mathbb{R}$). However, problems such as the problem of infinite in gravity are yet unsolved. To avoid such problems, quantum gravity theories like superstring theory or loop quantum gravity are developing, but neither of those theories has been completed. On the other hand, the assumption that all of the physics in the universe is constructed solely of $\mathbb{R}$ has not been proved. For instance, Yukawa considered extended wave function of elementary particles using "elementary domains" [1] and Snyder [2] and Finkelstein [3] simply considered quantization of space-time. However, these approaches are incomplete and a unified formulation is unknown. Here, we assert that a different mathematical structure, namely a world with a finite [4] and discrete space-time [5–14], is needed for the following reasons.

The description of a discrete space-time would require a discrete mathematical structure. Unitary transformation from quantum mechanics and Lorentz (Möbius) transformation from relativity requires this structure to be a number field, where addition, subtraction, multiplication, and division can be defined. The set of integers $\mathbb{Z}$ is not a field because division is not defined. For instance, fractional numbers such as 1/3 are not defined. What we would consider in this paper is called a Galois field ($\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$), that is, the set of integers where two numbers differing by a multiple of a prime number p are equivalent [5–14]. $\mathbb{F}_p$ has a finite, periodic, and discrete structure. Mathematical and physical justification to consider this field are as follows.

Mathematically, there is another system called p-adic numbers $\mathbb{Q}_p$ (p : prime) that is equivalent to $\mathbb{R}$ [15]. $\mathbb{Q}_p$ is regarded as the limit of p-adic integers $\mathbb{Z}/p^n\mathbb{Z}$ with $n \to \infty$.

$$\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}/p^3\mathbb{Z}, \cdots, \mathbb{Z}/p^n\mathbb{Z} \overset{n\to\infty}{\longrightarrow} \mathbb{Q}_p.$$

Moreover, In adele ring theory, $\mathbb{R}$ is regarded as the limit of $\mathbb{Q}_p$ with $p \to \infty$ [16].

$$\mathbb{Q}_2, \mathbb{Q}_3, \mathbb{Q}_5, \cdots, \mathbb{Q}_p \overset{p\to\infty}{\longrightarrow} \mathbb{R}.$$

So, from the viewpoint of number theory, the geometry constructed by the real numbers $\mathbb{R}$ is only an approximation of a finite and discrete geometry [11,17]. Physics over $\mathbb{R}$ may be obtained from physics over $\mathbb{F}_p$ under the appropriate limit.

Physically, Galois fields can appear naturally in our universe from multiple perspectives. First, recent observation of microwave background radiation suggests that the world is finite and periodic (I_{120}: Poincaré Dodecahedron) [4]. Second, if space-time has the Planck length as a characteristic scale, then Lorentz transformation must be discrete [18]. Theoretically, Jarnefelt estimated a huge but finite number of world coordinates as $10^{10^{81}}$ [5,7,19]. Third, the theory of quantum gravity needs to be expanded because of the difficulty of renormalizability. This problem would not appear in $\mathbb{F}_p$ in the first place [5,6,13,14], so the expansion is expected to be achieved by using a discrete system. In standard particle physics theory, the existence of antiparticles depends on additional assumptions, while in the Galois field such strange assumptions are not necessary [13]. Fourth, the field theory of Galois field was investigated by Nambu in the context of recurrence time [6]. Recently, the discreteness of time is also discussed the context of time crystals [20–24]. So, if space-time is both finite and discrete, we have to re-examine the laws of physics [6].

In this article, we re-examine a previous model and classify hadrons by using a discrete Galois field $\mathbb{F}_p$. When we reconstruct new theories with a Galois field, these new theories must satisfy fundamental conservation laws, namely those related to unitarity, Lorentz invariance, and gauge invariance. Consequently, instead of the Nakano–Nishijima–Gell-Mann (NNG) formula (which is proposed using continuous coordinate $\mathbb{R}$), we obtained the alternate formula $Q = 2(n + I)$, where Q is charge number, n is multivaluedness in Galois field, and I is total isospin. This formula is derived from Meson (two quarks) and Baryon (three quarks) representations in a unified way. The use of isospin was not investigated in detail by previous authors [5,7–12,19,25] who introduced finite geometry into the field theory. Since the NNG formula also applies to individual quarks, we apply the new formula to up-quark and down-quark, which inevitably lead to fractional numbers. Fractional numbers in Galois fields are, however, very large numbers. Therefore, isolated quarks with fractional charge can have a very large energy. We surmise that *quark confinement* ascribes to this fractionality in Galois field. These results may be a starting point to develop a theory without problems of infinity.

2. Galois Field

Here, we introduce the Galois fields $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$. A number field is a mathematical structure where addition, subtraction, multiplication and division are defined. Let $\mathbb{Z}/m\mathbb{Z}$ be the set of integers where two numbers differing by an integral multiple of $m \in \mathbb{Z}$ are regarded equivalent. A well-known result from number theory is that $\mathbb{Z}/m\mathbb{Z}$ is a number field if and only if m is a prime number. This finite field is called a Galois field which is denoted by $\mathbb{F}_p$:

$$\mathbb{F}_p = \{0, 1, 2, \ldots, p - 1\}. \tag{1}$$

A Galois field is not an ordered field. But, as was made clear in Reference [25], $\mathbb{F}_p$ can be partially ordered if p has the form

$$p = 8x \prod_{i=1}^{k} q_i - 1 \tag{2}$$

where x is an odd integer and $\prod_i q_i$ is the product of the first k odd primes. So, the usual notion of magnitude (large and small) and sign (positive and negative) hold for the first $N \sim q_k \sim \ln p$ elements. Therefore, we consider a prime of the form in Equation (2) which is assumed to be large enough to construct a geometry containing all coordinates in the universe.

Quantum mechanics requires complex numbers. To consider complex numbers in a Galois field, it is natural to take an analogue of the ordinary extension of real numbers to complex numbers. That is to say, the set of "complex" numbers

$$z = x + iy, \quad (x, y \in \mathbb{F}_p) \tag{3}$$
$$i^2 = -1 \tag{4}$$

is a "complex" Galois field which is denoted by $\mathbb{F}_{p^2}$. For a prime p of the form $p = 4n - 1$ ($n \in \mathbb{Z}$), complex conjugation is given by $i^p = -i = i^*$ and $z^p = x - iy = z^*$ since we have

$$(x + iy)^p = x^p - iy^p = x - iy. \tag{5}$$

The first equality holds because p is a character of the field, that is $p \cdot a = 0$ for all $a \in \mathbb{F}_p$; The second equality follows from Fermat's little theorem $a^p \equiv a \mod p$. Note that $|z|^2$ can become negative for the non-ordered part of $\mathbb{F}_p$.

3. Lorentz Group in a Galois Field: The Coish Group

Lorentz transformation can be recovered with the Galois field introduced in the previous section [5,8,11,12]. Coordinates in Minkowski space-time are closely related to spinor structure through the Pauli matrix representation

$$X = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} = x_0\mathbf{1} + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 \tag{6}$$

where $x_0, x_1, x_2, x_3 \in \mathbb{F}_p$ and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Pauli matrices. (See Reference [26] for the case of $x_\mu \in \mathbb{R}$.) The metric form is given as the determinant of X:

$$\|X\| = x_0^2 - x_1^2 - x_2^2 - x_3^2. \tag{7}$$

Let a be a 2×2 matrix with coefficients in $\mathbb{F}_{p^2}$. The linear transformation

$$X' = a^\dagger X a \tag{8}$$

is a proper Lorentz transformation if it leaves the metric form Equation (7) invariant, that is, if $\|a\|^*\|a\| = \|a\|^{p+1} = 1$ holds. Such matrices form the orthogonal group $\mathrm{SL}(2, \mathbb{F}_{p^2})$. In finite geometry, however, there also exist transformations with $\|a\|^{p+1} = -1$ which reverse the sign of the metric [5].

In addition, suppose that matrices a_1 and a_2 lead to the same Lorentz transformation. Then $U \equiv a_1 a_2^{-1}$ transforms X as

$$U^\dagger X U = X. \tag{9}$$

Since the set of X contains the identity matrix, Equation (9) leads to $U^\dagger U = UU^\dagger = 1$. So, U is a unitary matrix of the form

$$U = \lambda\mathbf{1}. \tag{10}$$

Given that U is unitary, λ satisfies

$$\lambda^* \lambda = 1. \tag{11}$$

In a Galois field, such λ become discrete and multivalued. That is, for a prime p of the form $p = 4n - 1$ (such as in Equation (2)) we have

$$\lambda = \omega^\alpha \ , \ a_1 = \omega^\alpha a_2. \tag{12}$$

where $\alpha = 0, \ 1, \ \cdots , \ p$. So, the the orthogonal group in a Galois field has $p + 1$ multivaluedness as with a cyclotomic field, and ω acts as a phase factor (Figure 1). In this way, we obtain the full Coish group. The spinor group, which is a subgroup of the rotational group, has the same multivaluedness. This multivaluedness will plays an essential role for the classification of hadrons.
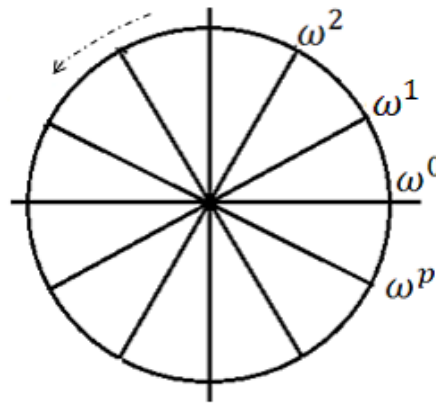

Figure 1. Schematic representation of the discrete multivaluedness with phase factor ω. Each set of $\mathbb{F}_{p^2}$ constitutes a cyclotomic field. When $p = 4n - 1$, the field has $p + 1$ elements. A Lorentz group in a Galois field has $p + 1$ multivaluedness such as $\omega^\alpha a$, $\alpha = 0, 1, \cdots , p$. Regarding ω as a phase factor, it constitute a global gauge transformation in a Galois field.

4. Gauge Transformation in Galois Field

In this section, the charge number Q is derived with gauge transformation in Galois field in accordance with the derivation given in Reference [5]. Because of homomorphism, an orthogonal group leads a spinor group. The irreducible representation of the orthogonal group must be a matrix with elements of $\mathbb{F}_{p^2}$ as its coefficients. The spinor group is a subgroup of of the two dimensional general linear homogeneous Galois group $GLH(2, p^2)$ over the complex finite field $\mathbb{F}_{p^2}$. The irreducible representations of $GLH(2, p^2)$ are given by Brauer and Nesbitt [27] and extended by Coish [5] as

$$\|a\|^n a^{(2j)} \otimes \bar{a}^{(2k)}, \tag{13}$$

where

$$j, k = 0, \ 1/2, \ 1, \ 3/2, \cdots (p - 1)/2, \tag{14}$$

$$n = 0, \ \pm 1/2, \ \pm 1, \cdots \pm (p + 1/2), \ p + 1. \tag{15}$$

Here, a is a spinor matrix which transforms a spinor

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \psi_1' \\ \psi_2' \end{pmatrix} = a \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{16}$$

$\|a\|^n$ is interpreted as the multivaluedness of the representation. Next, consider the polynomial

$$f(\psi) = \sum_{m=-j}^{j} f_m \psi_1^{j+m} \psi_2^{j-m}. \tag{17}$$

The set $\left\{ \psi_1^{j+m} \psi_2^{j-m} \right\}$ can be regarded as the linearly independent set of basis for a $(2j+1)$-dimensional vector space. The action of a on $f(\psi)$ is given by

$$f(a\psi) = \sum_{m=-j}^{j} f_m (\psi_1')^{j+m} (\psi_2')^{j-m} = \sum_{m=-j}^{j} f_m a_{mn}^{(2j)} \psi_1^{j+n} \psi_2^{j-n}. \tag{18}$$

So, $a^{(2j)}$ is a matrix transforming the monomials $\psi_1^{j+m} \psi_2^{j-m}$. In particular, the phase factor ω^α transforms $a \to \omega^\alpha a$ and $a^{(2j)} \to \omega^{2j\alpha} a^{(2j)}$. This phase factor corresponds to a gauge transformation, and Equation (13) is transformed as follows

$$\|a\|^n a^{(2j)} \otimes \bar{a}^{(2k)} \longrightarrow \omega^{Q\alpha} \|a\|^n a^{(2j)} \otimes \bar{a}^{(2k)}, \tag{19}$$

where

$$Q = 2(n + j - k). \tag{20}$$

Therefore, we naturally derive Equation (20) as the charge number reflecting discreteness of gauge transformation in Galois field.

5. Classification of Hadrons

Now, we can classify Hadrons in a Galois field. Coish interpreted j and k as spins and represented baryons and mesons with Equations (13) and (20). However, his model does not agree with the standard model of particle physics. So, we give a different interpretation. Instead of spins, we interpret j and k as isospin. More precisely, to encompass the multiplet structure of hadrons, it would be appropriate to interpret j, k as the total isospin I instead of I_z. Mesons, which are composed of particles and anti-particles, are transformed by the representation

$$D_{\text{meason}}^{(j,-k,n)} \equiv \|a\|^n a^{(2j)} \otimes \bar{a}^{(2k)}, \tag{21}$$

but it is now interpreted as a representation of $GLH(2, p^2) \times GLH(2, p^2)^*$. Baryons, which are composed of three quarks, are transformed by another representation

$$D_{\text{baryon}}^{(i,j,k,n)} \equiv \|a\|^n a^{(2i)} \otimes a^{(2j)} \otimes a^{(2k)} \tag{22}$$

of $GLH(2, p^2) \times GLH(2, p^2) \times GLH(2, p^2)$. Here, the contribution of strangeness appears as a modification of the multivaluedness n.

Mesons and baryons can be classified according to the charge number

$$Q = 2(n + I), \tag{23}$$

where $I = j - k$ for mesons and $I = i + j + k$ for baryons.

Tables 1 and 2, respectively, show quantum numbers and representations of mesons and baryons in a Galois field. Here, we did not use $SU(3)$ symmetry but assume the known quark contents.

Table 1. Quantum numbers of mesons in a Galois field. Q is the charge number of baryon and j, k are total isospins of quarks. n is derived from Equation (23).

Meson	Q	n	I	j	k	Representation
K^+	1	0	1/2	1/2	0	$\|a\|^0 a^{(1)} \otimes \bar{a}^{(0)}$
K^0	0	$-1/2$	1/2	1/2	0	$\|a\|^{-\frac{1}{2}} a^{(1)} \otimes \bar{a}^{(0)}$
π^+	1	1/2	0	1/2	1/2	$\|a\|^{\frac{1}{2}} a^{(1)} \otimes \bar{a}^{(1)}$
π^0	0	0	0	1/2	1/2	$\|a\|^0 a^{(1)} \otimes \bar{a}^{(1)}$
π^-	-1	$-1/2$	0	1/2	1/2	$\|a\|^{-\frac{1}{2}} a^{(1)} \otimes \bar{a}^{(1)}$
$\bar{K}^0$	0	1/2	$-1/2$	0	1/2	$\|a\|^{\frac{1}{2}} a^{(0)} \otimes \bar{a}^{(1)}$
K^-	-1	0	$-1/2$	0	1/2	$\|a\|^0 a^{(0)} \otimes \bar{a}^{(1)}$

Table 2. Quantum numbers of baryons in a Galois field. Q is the charge number of baryon and i, j, k are total isospins of quark. n is derived from Equation (23).

Baryon	Q	n	I	i	j	k	Representation
p	1	-1	3/2	1/2	1/2	1/2	$\|a\|^{-1} a^{(1)} \otimes a^{(1)} \otimes a^{(1)}$
n	0	$-3/2$	3/2	1/2	1/2	1/2	$\|a\|^{-\frac{3}{2}} a^{(1)} \otimes a^{(1)} \otimes a^{(1)}$
Σ^+	1	$-1/2$	1	1/2	1/2	0	$\|a\|^{-\frac{1}{2}} a^{(1)} \otimes a^{(1)} \otimes a^{(0)}$
Σ^0	0	-1	1	1/2	1/2	0	$\|a\|^{-1} a^{(1)} \otimes a^{(1)} \otimes a^{(0)}$
Σ^-	-1	$-3/2$	1	1/2	1/2	0	$\|a\|^{-\frac{3}{2}} a^{(1)} \otimes a^{(1)} \otimes a^{(0)}$
Ξ^0	0	$-1/2$	1/2	1/2	0	0	$\|a\|^{-\frac{1}{2}} a^{(1)} \otimes a^{(0)} \otimes a^{(0)}$
Ξ^-	-1	-1	1/2	1/2	0	0	$\|a\|^{-1} a^{(1)} \otimes a^{(0)} \otimes a^{(0)}$

We derived Equation (23) equivalent to the NNG formula

$$Q = I_z + \frac{Y}{2}, \tag{24}$$

where Y is the hypercharge. Figure 2 shows the classification of hadrons with n and I. Particles and anti-particles are related by CP conjugation (that is, point symmetry around the origin $n = I = 0$) as in Reference [5]. Different values of I indicate the multiplet structure of hadrons. By comparison with the NNG formula, n includes both the isospin I_z and the hypercharge Y. In this way, Galois fields can be used to organize the quantum number of hadrons. Similarly, as it is well known, the quantum number of leptons are related with an equation similar to the NNG formula; the origins of the weak isospin and weak hypercharge, which are properties of leptons introduced phenomenologically, may also be explained by the Galois field.
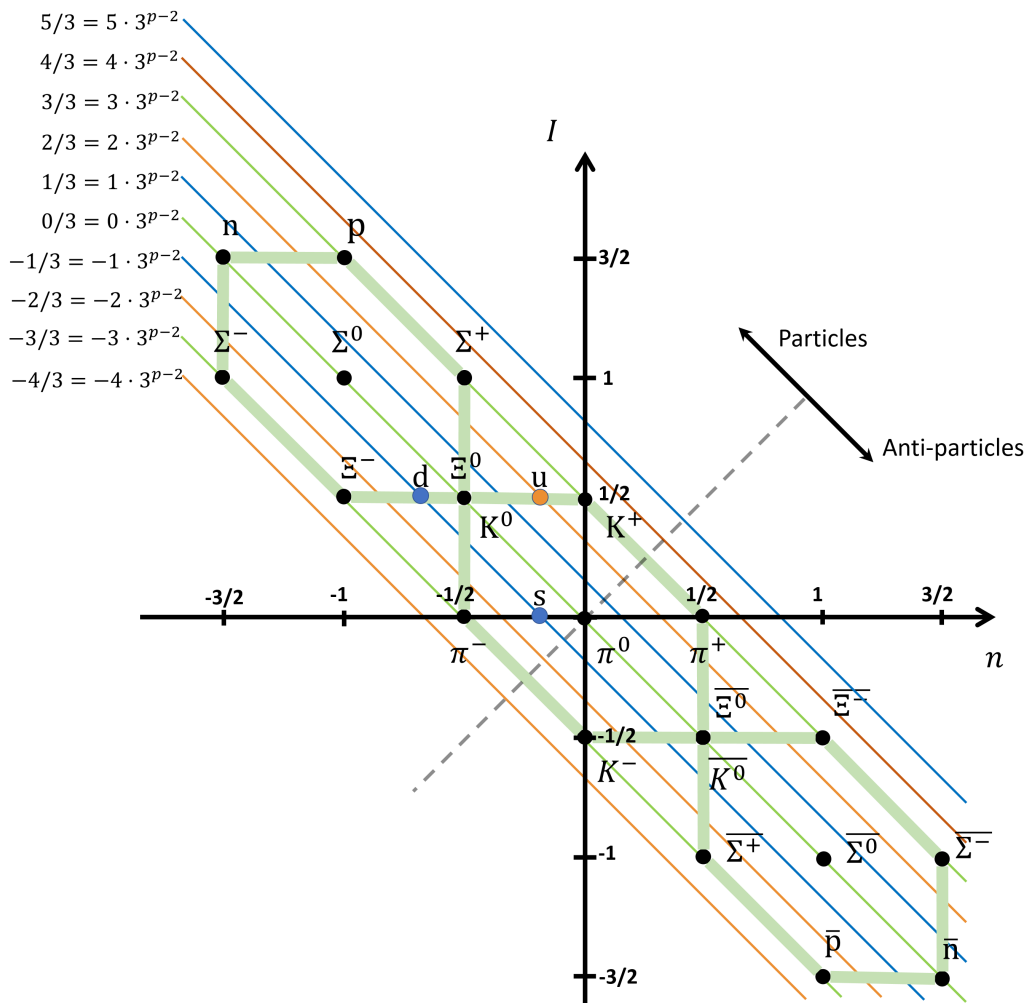
Figure 2. Classification of hadrons with Equation (23). Our model take the n and I axis instead of Y and I_z. We did not use SU(3) symmetry but assume the same quark constituent as in the standard model of particle physics. Charge numbers such as $Q = 2/3 = 2 \cdot 3^{p-2}$ mod p are shown in Figure 3. Note that this figure superimposes baryons, anti-bayons, and mesons, which differ by baryon numbers. So, K^0 and Ξ^0, for instance, are distinguished.

More generally, we can give a unified representation for arbitrary composite particles. First, we can give a representation of a single particle as $\|a\|^n a^{(2j)}$. Likewise, a representation of particles with $N \, (= 2, 3, 4, 5$ or higher) constituents is possible with

$$\bigotimes_{l=1}^{N} \|a\|^{n_l} a^{(2j_l)\theta_{v_l}} \tag{25}$$

where j_l are half-integers, $v_l = 0$ or 1, θ_0 is the identity map $a^{(2j)\theta_0} = a^{(2j)}$, and θ_1 maps $a^{(2j)}$ to the conjugate group $a^{(2j)\theta_1} = \bar{a}^{(2j)}$ with coefficients $\alpha^{p^1} = a^*$ (c.f. References [5,27]). Then, the charge number is given by $Q = \sum_{l=1}^{N} 2(n_l \pm j_l)$ where plus and minus correspond to $v_l = 0$ and $v_l = 1$, respectively. This includes a representation of mesons, baryons, pentaquark, and so on, in a unified way.

6. Quark Confinement

Finally, we provide a possible approach toward quark confinement which ascribes to fractionality in Galois field.

Suppose for the moment that up-quark and down-quark can be described by the following irreducible representation of $GLH(2, p^2)$:

$$a_u = \|a\|^{n_u} a^{(1)}, \qquad a_d = \|a\|^{n_d} a^{(1)}. \tag{26}$$

We expect that proton and neutron can be expressed as $a_p = a_u \otimes a_u \otimes a_d$ and $a_n = a_u \otimes a_d \otimes a_d$, respectively. Then, the quantum numbers of proton and neutron can be written in terms of their constituent quarks as follows:

$$n_p = 2n_u + n_d, \tag{27}$$

$$n_n = n_u + 2n_d, \tag{28}$$

$$I_p = I_n = I. \tag{29}$$

Substitute this decomposition into Equation (23) and we obtain

$$Q_p = 2(2n_u + n_d + I), \quad Q_n = 2(n_u + 2n_d + I). \tag{30}$$

Then, we can solve for n_u and n_d

$$n_u = \frac{2Q_p - Q_n - 2I}{6}, \quad n_d = \frac{2Q_n - Q_p - 2I}{6}. \tag{31}$$

Taking the values $Q_p = 1$, $Q_n = 0$ and $I = 3/2$ as in Table 2 we obtain

$$n_u = -\frac{1}{6}, \quad n_d = -\frac{2}{3}, \tag{32}$$

which are not half-integers as in (15). Therefore, irreducible representations of quarks as in Equation (26) is not possible.

Now, suppose that we can nevertheless write the corresponding charge numbers as elements of $\mathbb{F}_p$. Then, we obtain $Q_u = 2/3$ and $Q_d = -1/3$. Counterintuitively, these fractional quantum numbers in $\mathbb{F}_p$ are very large, which can be shown as follows. The modular inverses in of an integer a in $\mathbb{F}_p$ is defined by

$$a^{-1} a \equiv 1 \bmod p. \tag{33}$$

On the other hand, from Fermat's little theorem

$$a^{p-1} \equiv 1 \bmod p \tag{34}$$

we obtain

$$a^{-1} \equiv a^{p-2} \bmod p. \tag{35}$$

The prime given by Equation (2) contains the first odd prime $q_1 = 3$, so it has the form $p = 3n - 1$ for some $n \in \mathbb{N}$. In this case, the fractional number $1/3$ in $\mathbb{F}_p$ takes the value $3^{-1} = (p+1)/3$ (see Figure 3). Therefore, fractional numbers in Galois field are large numbers of order p. Consequently, isolated quark can have, for instance a *huge* electromagnetic self-energy (Figure 4). In this case, quark confinement in $\mathbb{F}_p$ may be a consequence of energy minimization.
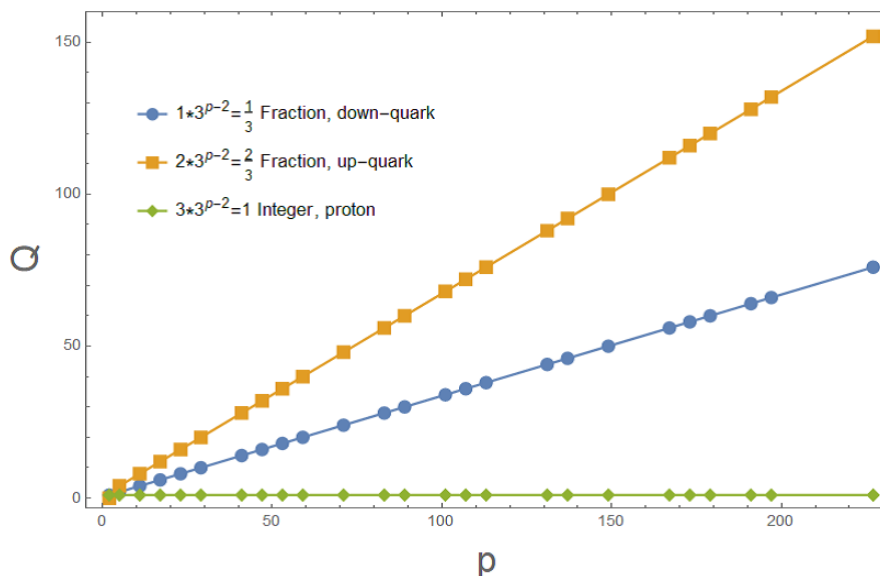
Figure 3. The values of $1/3 = 3^{p-2} \bmod p = (p+1)/3$ and $2/3 = 2 \cdot 3^{p-2} \bmod p = 2(p+1)/3$ are shown for the primes of the form $p = 3n - 1$. Fractional numbers in a Galois field are large numbers proportional to p.
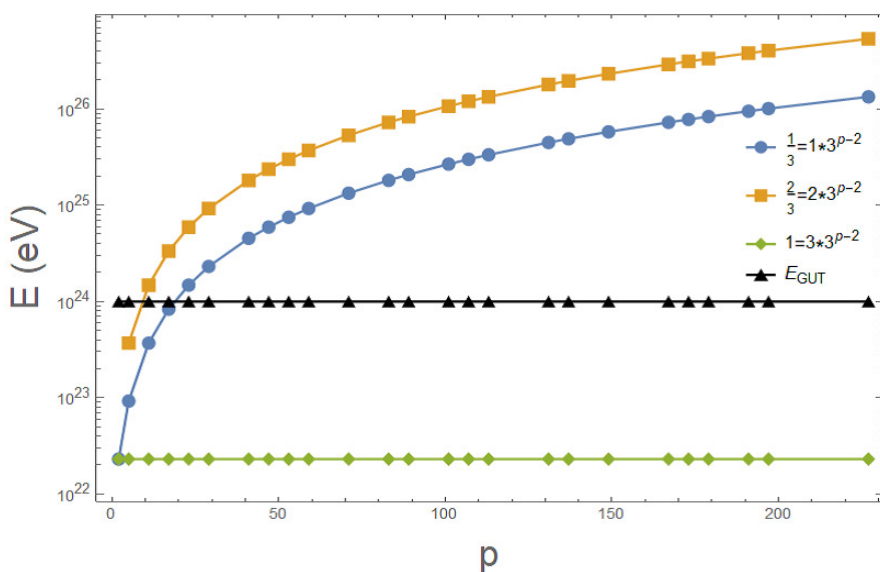

Figure 4. Electromagnetic self-energy $E_{\text{em}} = k_e \frac{Q^2}{2a} c^2$ where Q is the charge number in Galois field, k_e is the Coulomb constant, a is the classical electron radius, and c is the speed of light. The energy is independent of p for integral charge but increase for fractional charge. The GUT energy is shown as a rough standard.

7. Discussion and Conclusions

Finally, we summarize and discuss our results. The NNG formula is a phenomenological equation. By using a Galois field, the NNG Formula (24) is rewritten as (23). The new quantum number n, which includes isospin I_z and hypercharge Y, arises from the multivaluedness of a linear representation of the Galois field. We also discussed quark confinement which ascribes to fractionality of Galois field. After all, these results may indicate finiteness and discreteness of the world. While such a method is unconventional, it is not inherently inconsistent under one assumption, namely that the geometry constructed by the real numbers $\mathbb{R}$ is only an approximation of a finite and discrete geometry [11,17].

As future developments, we expect that composite particles such as pentaquarks [28–30] can be explained by generalizing Equation (25) to any number of particles and combining the quantum numbers n and I to a single "grand spin" as discussed in [30]. We expect that the inclusion of other flavors such as c (charm), t (top) and b (bottom) would require extension of our result to $GLH(3, p^2)$ (analogous to SU(3)), $GLH(4, p^2)$, and so on, which goes in parallel to the extension of SU(2) to SU(3) or higher. Representation of particles in Figure 2 can be modified accordingly. This extention can be done based on previous studies [31] and using a method equivalent to the "enlarging lemma" [32]. Recently, with the development of quantum information, Galois fields have begun to be used to construct space-time models as error-correcting codes [33]. Further application of Galois fields is expected in the foundation of quantum physics [34] and time crystal with discrete space-time symmetry [20–24].

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