Quantum time crystal by decoherence: Proposal with an incommensurate charge density wave ring

K. Nakatsugawa,[1] T. Fujii,[3] and S. Tanda[1,2]

[1]*Department of Applied Physics, Hokkaido University, Sapporo 060-8628, Japan*
[2]*Center of Education and Research for Topological Science and Technology, Hokkaido University, Sapporo 060-8628, Japan*
[3]*Department of Physics, Asahikawa Medical University, Asahikawa 078-8510, Japan*

We show that time translation symmetry of a ring system with a macroscopic quantum ground state is broken by decoherence. In particular, we consider a ring-shaped incommensurate charge density wave (ICDW ring) threaded by a fluctuating magnetic flux: the Caldeira-Leggett model is used to model the fluctuating flux as a bath of harmonic oscillators. We show that the charge density expectation value of a quantized ICDW ring coupled to its environment oscillates periodically. The Hamiltonians considered in this model are time independent unlike "Floquet time crystals" considered recently. Our model forms a metastable quantum time crystal with a finite length in space and in time.

I. INTRODUCTION

The original proposal of a quantum time crystal (QTC) given by Wilczek [1] and Li *et al.* [2] is a quantum mechanical ground state which breaks time translation symmetry. In this QTC ground state there exists an operator $\hat{Q}$ whose expectation value oscillates permanently with a well-defined "lattice constant" P, that is, with a well-defined period. Volovik [3] relaxed the condition of permanent oscillation and proposed the possibility of effective QTC, that is, a periodic oscillation in a metastable state such that the oscillation will persist for a finite duration $\tau_Q \gg P$ in the time domain and will eventually decay.

In this paper we promote Volovik's line and consider the possibility of a metastable QTC state without spontaneous symmetry breaking: We consider symmetry breaking by *decoherence* of a macroscopic quantum ground state (Fig. 1). Decoherence is defined as the loss of quantum coherence of a system coupled to its environment. Coupling to environment will inevitably introduce friction to the system such that the oscillation will eventually decay at $t = \tau_{\text{damp}}$. However, for $t < \tau_Q \ll \tau_{\text{damp}}$ the oscillation period P is well defined. If friction is sufficiently weak such that $P \ll \tau_Q$, then we have a model of effective QTC with lifetime τ_Q.

Our model consists of a ring-shaped incommensurate charge density wave (ICDW ring) threaded by a fluctuating magnetic flux [Fig. 2(a)]. A charge density wave (CDW) is a periodic (spatial) modulation of electric charge density which occurs in quasi-one-dimensional crystals [5]: the periodic modulation of the electric charge density occurs due to electron-phonon interaction. If the ratio between the CDW wavelength λ and the lattice constant a of the crystal is a simple fraction like 2, 5/2, etc., then the CDW is commensurate with the underlying lattice. A commensurate CDW cannot move freely because of commensurability pinning; i.e., the CDW phase is pinned by the ion's position in the crystal. On the other hand, if λ/a is effectively an irrational number [5], then the CDW is incommensurate with the underlying lattice. An ICDW ring with a radius of 10 μm, for instance, contains approximately 10^5 wavelengths [6], so it is possible that a/λ is very close to an irrational number. See the discussion section for an elaboration of this assumption. The sliding of an ICDW without pinning is described by a gapless Nambu-Goldstone (phason) mode [5] and the energy of an ICDW is

independent of its phase (i.e., position), which implies that the expected ground state of an ICDW ring is a superposition of ICDWs with different phases. Ring-shaped crystals and ring-shaped (I)CDWs have been produced [7] [Fig. 2(b)]. The presence of circulating CDW current [8] and Aharonov-Bohm oscillation (evidence of macroscopic wave function) [9] is verified experimentally.

We show in Sec. II that the charge density expectation value of an isolated ICDW ring with moment of inertia I is periodic in time with period $P = 4\pi I/\hbar$. This periodicity is a consequence of the uncertainty relation on the S^1 (ring). However, this oscillation becomes unobservable at ground state because the ground state of an isolated ICDW ring is a plane wave state, i.e., a coherent superposition of ICDWs with different phases. Therefore, in Sec. III we use the Caldeira-Leggett model [10,11] to show that time translation symmetry is broken by decoherence. More precisely, the superposition is broken by decoherence and the amplitude of the ICDW oscillates periodically [Fig. 2(c)]. If the ICDW ring weakly couples to its environment then this state is a metastable ground state. Therefore, our model forms an effective QTC with a finite length in space and in time.

Before developing our main arguments, we compare our work to recent developments of QTC. In analogy with spatial crystals, the original proposal of a QTC is based on the spontaneous breaking of time translation symmetry. However, Bruno [12] and Watanabe and Oshikawa [13] theoretically proved that spontaneous breaking of time translation symmetry cannot occur at ground state. Recently, it was shown that there is a notion of spontaneous breaking of time translation symmetry in periodically driven (Floquet) states [14] and this idea was proved experimentally [15]. On the other hand, the periodic oscillation we consider in this paper is inherent to ring systems with a macroscopic wave function.

II. GROUND STATE OF AN ISOLATED ICDW RING

A. Classical theory of ICDW ring

It is well known that the electric charge density of a quasi-one-dimensional crystal becomes periodic by opening a gap at the Fermi wave number k_F and forms a charge density wave (CDW) ground state with a wavelength $\lambda = \pi/k_F$
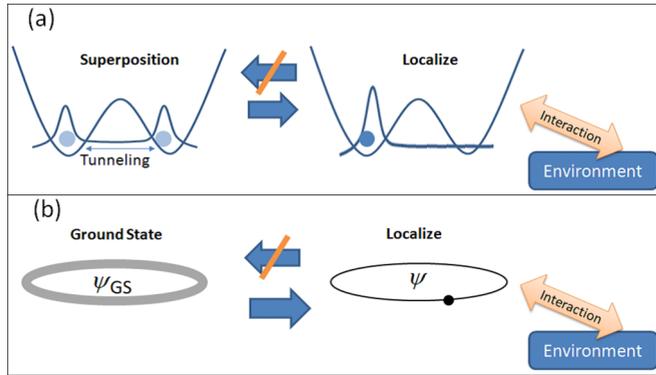
FIG. 1. The concept of translation symmetry breaking by decoherence is illustrated. (a) A simplified version of the two-state system considered by Leggett *et al.* [4]. A particle can tunnel through a potential barrier and exist at two states simultaneously. However, if this system starts to interact with its surrounding environment, then the particle will localize at one of the states. (b) Similarly, the ground state of a free particle confined on a ring is a plane wave state. Coupling to environment will localize the particle and break rotational symmetry. This "particle" corresponds to the phase of an incommensurate charge density wave ring (ICDW ring) in our model (Fig. 2).

[5]. Consider a CDW formed on a ring-shaped quasi-one-dimensional crystal with radius R. The order parameter of this CDW ring is a complex scalar $\Delta(x,t) = |\Delta(x,t)| \exp[i\theta(x,t)]$, where $|\Delta(x,t)|$ is the size of the energy gap at $\pm k_F$, $\theta(x,t)$ is the phase of the CDW, $x \in [0, 2\pi R)$ is the coordinate on the crystal, and t is the time coordinate. The charge density is given by

$$n(x,t) = n_0 + n_1 \cos[2k_F x + \theta(x,t)], \qquad (1)$$

where n_0 is the average charge density and n_1 is the amplitude of the wave. Bogachek *et al.* [16] derived the following Lagrangian density of the phase of a ring-shaped incommensurate CDW (ICDW ring) threaded by a magnetic flux

$$\mathcal{L}_0\left(\frac{\partial \theta}{\partial t}, \frac{\partial \theta}{\partial x}\right) = \frac{N_0}{2}\left[\left(\frac{\partial \theta}{\partial t}\right)^2 - c_0^2\left(\frac{\partial \theta}{\partial x}\right)^2\right] + \frac{eA}{\pi}\frac{\partial \theta}{\partial t},$$

where A is the magnetic vector potential, $N_0 = v_F^2 \hbar^2 N(\varepsilon_F)/(2c_0^2)$, $N(\varepsilon_F)$ is the density of states of electrons at the Fermi level per unit length and per spin direction, v_F is the Fermi velocity of the crystal, $c_0 = \sqrt{m/m^*} v_F$ is the phason velocity, m^* is the effective mass of electrons, and $\hbar$ is the reduced Planck constant. We first consider an isolated ICDW ring with $A = 0$. Assuming that $N(\varepsilon_F)$ is equivalent to the density of states of electrons on a one-dimensional line, that is, $N(\varepsilon_F) = 1/(\pi \hbar v_F)$, we have $N_0 = \hbar v_F/(2\pi c_0^2)$. An incommensurate CDW (ICDW) can slide freely because of spatial translation symmetry, so the dynamics of an ICDW is understood by its phase θ. We further assume the rigid-body model of ICDW; i.e., the ICDW ring does not deform locally and the phase $\theta(x,t) = \theta(t)$ is independent of position. Then, the Lagrangian, the canonical angular momentum, and the
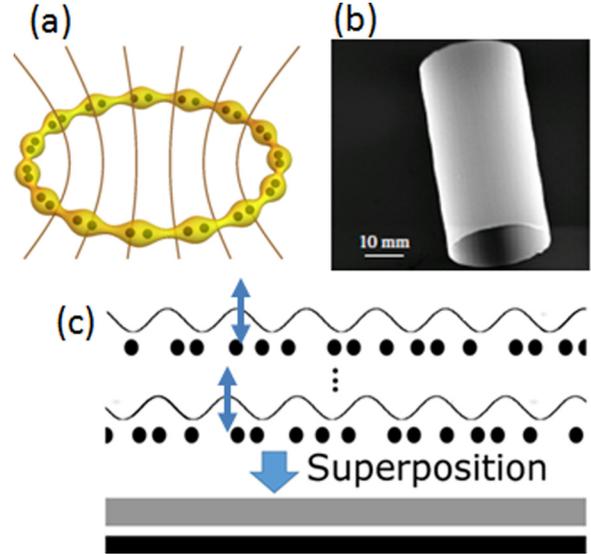

FIG. 2. (a) We consider an incommensurate CDW ring threaded by a fluctuating magnetic flux. This figure shows a commensurate CDW with $\lambda/a = 2$ because it is easier to visualize. The wave (typically $\sim 10^5$ wavelengths) represents the charge density and the dots represent the atoms of a quasi-one-dimensional crystal. (b) ICDW ring crystals such as monoclinic TaS$_3$ ring crystals and NbSe$_3$ ring crystals can be produced experimentally [7–9]. Our model can be tested provided clean ring crystals with almost no defects and impurities can be produced. (c) The ground state of an isolated quantized ICDW ring is a superposition of periodically oscillating ICDWs; hence the oscillation is unobservable. Coupling to environment (fluctuating magnetic flux) will break the superposition and the oscillation becomes apparent.

Hamiltonian of the ICDW ring are, respectively,

$$L_0(\dot{\theta}) = \int_0^{2\pi R} dx \, \mathcal{L}_0(\dot{\theta}) = \frac{I}{2}\dot{\theta}^2, \qquad (2)$$

$$\pi_\theta(\dot{\theta}) = \frac{\partial L_0(\dot{\theta})}{\partial \dot{\theta}} = I\dot{\theta}, \qquad (3)$$

$$H_0(\pi_\theta) = \pi_\theta \dot{\theta} - L_0(\dot{\theta}) = \frac{\pi_\theta^2}{2I}, \qquad (4)$$

where $\dot{\theta} = d\theta/dt$ and $I = \hbar R v_F/c_0^2$ is the moment of inertia. We note that (2), (3), and (4) are time independent.

B. Quantization of an isolated ICDW ring

Next, we quantize the ICDW ring system. We show that a quantized ICDW ring possesses an inherent oscillation which originates from the uncertainty principle. Let $\hat{H}_0 = \hat{\pi}_\theta^2/(2I)$ and $\hat{\pi}_\theta$ be the Hamiltonian operator and angular momentum operator of the ICDW ring, respectively. The macroscopic quantum state $\psi \in \mathcal{H}$ is defined in the Hilbert space $\mathcal{H}$ of positive square-integrable functions with the periodic boundary condition $\psi(\theta + 2\pi) = \psi(\theta)$. The canonical commutation relation $[\hat{\theta}, \hat{\pi}_\theta] = i\hbar$ is not satisfactory because $\hat{\theta}$ is a multivalued operator and is not well defined. Ohnuki and Kitakado [17] resolved this difficulty by using the unitary operator $\hat{W}$ and the self-adjoint angular momentum operator

$\hat{\pi}_\theta$ defined by

$$\langle\theta|\hat{W}|\psi\rangle = e^{i\theta}\psi(\theta), \quad \langle\theta|\hat{\pi}_\theta|\psi\rangle = -i\hbar\frac{\partial\psi(\theta)}{\partial\theta},$$

which satisfy the commutation relation on $\mathcal{H}$

$$[\hat{\pi}_\theta, \hat{W}] = \hbar\hat{W}. \tag{5}$$

$\hat{H}_0$ is a function of $\hat{\pi}_\theta$ only; hence the complete orthonormal set $\{\psi_l\}_{l=-\infty}^{\infty}$ of momentum eigenstates spans $\mathcal{H}$ and satisfies $\psi_l(\theta) = e^{il\theta}/\sqrt{2\pi}$. The eigenvalues of $\hat{\pi}_\theta$ are quantized with $\langle\psi_l|\hat{\pi}_\theta|\psi_l\rangle = l\hbar, l \in \mathbb{Z}$. $\hat{W}$ and $\hat{W}^\dagger$ are ladder operators which satisfy $\hat{W}\psi_l = \psi_{l+1}$ and $\hat{W}\psi_l = \psi_{l-1}$. Therefore, (5) is the one-dimensional version of the well-known angular momentum algebra [18]. Time evolution is introduced via the Heisenberg picture: $\hat{\pi}_\theta(t) = e^{i\hat{H}_0 t/\hbar}\hat{\pi}_\theta e^{-i\hat{H}_0 t/\hbar}$ and $\hat{W}(t) = e^{i\hat{H}_0 t/\hbar}\hat{W}e^{-i\hat{H}_0 t/\hbar}$. $\hat{\pi}_\theta$ commutes with $\hat{H}_0$, so $\hat{\pi}_\theta(t) = \hat{\pi}_\theta$. From the commutation relation (5) we obtain the following solutions of $\hat{W}(t)$:

$$\hat{W}(t) = e^{it\hat{\pi}_\theta/I}\hat{W}e^{-\frac{it}{2\mu}} = \hat{W}e^{it\hat{\pi}_\theta/I}e^{\frac{it}{2\mu}}, \tag{6}$$

where $\mu = I/\hbar$. The two different expressions in (6) arise from the noncommutativity between $\hat{W}$ and $e^{it\hat{\pi}_\theta/I}$. For a QTC we need a periodic expectation value at the ground state. So, we define the time-dependent charge density operator

$$\hat{n}(x,t) = n_0 + \frac{n_1}{2}[e^{2ik_F x}\hat{W}(t) + \text{H.c.}] \tag{7}$$

and replace the classical charge density (1) by the expectation value

$$n(x,t) = \langle\hat{n}(x,t)\rangle. \tag{8}$$

Any states in $\mathcal{H}$ must be a linear superposition of $\{\psi_l\}$, that is $\psi = \sum_{l\in\mathbb{Z}} c_l \psi_l$ provided $\sum_l |c_l|^2 = 1$. Therefore, the expectation values $\langle\hat{W}(t)\rangle$ and $\langle\hat{n}(x,t)\rangle$ are periodic with period $P = 4\pi\mu$ for any state ψ:

$$\begin{aligned}\langle\hat{W}(t)\rangle &= \frac{1}{2}\text{tr}[\hat{W}(t)\hat{\rho} + \hat{\rho}\hat{W}(t)]\\ &= \frac{1}{2}\int_{-\pi}^{\pi} d\theta e^{i\theta}[e^{\frac{it}{2\mu}}\rho(\theta+t/\mu,\theta)\\ &\quad + \rho(\theta,\theta-t/\mu)e^{-\frac{it}{2\mu}}].\end{aligned} \tag{9}$$

From the two different expressions of $W(t)$ in (6) we can define the Weyl form of the commutation relation [19]

$$\hat{W}e^{it\hat{\pi}_\theta/I} = e^{it\hat{\pi}_\theta/I}\hat{W}e^{-\frac{it}{\mu}};$$

hence the phase $\frac{t}{2\mu}$ and the periodic oscillation with period $4\pi\mu$ is a manifestation of the uncertainty principle. For an alternative explanation, let us consider an electron with effective mass m^* confined in a finite space with volume $L \sim 2R$. From the uncertainty principle, the momentum uncertainty of this particle is $\Delta p \sim \hbar/(4R)$. This means that the particle's wave packet expands with velocity $v = \Delta p/m^* \sim \hbar/(4m^*R)$. Then, because of the periodic boundary condition, the physical quantity $W = e^{ix/\lambda}$ is periodic with period $P = \lambda/v \sim 4\pi m^*R/(\hbar k_F) = 4\pi m^*R/mv_F = 4\pi Rv_F/c_0^2 = 4\pi\mu$. So, the origin of the periodicity is that (i) the macroscopic wave function of the ICDW ring diffuses due to the uncertainty principle, and then (ii) $W = e^{i\theta}$ oscillates periodically. However, the oscillation in (9) is not observable at the ground state $\hat{\rho}_0 \equiv |\psi_0\rangle\langle\psi_0|$ because $\langle\theta|\hat{\rho}_0|\phi\rangle = \frac{1}{2\pi}$ and the θ integral vanishes. Therefore, the ground state of an isolated ICDW ring is not yet a time crystal because of superposition.

III. COUPLING TO ENVIRONMENT

Now, suppose that the ICDW ring starts to interact with its surrounding environment at $t = 0$. Then, we expect decoherence of the phase θ. This interaction is modeled using the Caldeira-Leggett model [10] which is a model quantum Brownian motion. It describes a particle coupled to its environment. This environment is described as a set of noninteracting harmonic oscillators. First, the classical solution $\theta(t)$ is calculated to study the dynamics of the ICDW ring. Next, this system is quantized to calculate the amplitude of the charge density expectation value $\langle\hat{n}(x,t)\rangle$.

A. Classical theory of ICDW ring with environment

Let us consider the following Lagrangian of an ICDW ring threaded by a fluctuating magnetic flux:

$$\tilde{L}(\dot{\theta},\mathbf{q},\dot{\mathbf{q}}) = \frac{1}{2}I\dot{\theta}^2 + A(\mathbf{q})\dot{\theta} + \sum_{j=1}^{\mathcal{N}}\left(\frac{1}{2}m\dot{q}_j^2 - \frac{1}{2}m\omega_j^2 q_j^2\right), \tag{10}$$

where q_j are the normal coordinates of the fluctuation and $\tilde{\pi}_\theta = \partial\tilde{L}/\partial\dot{\theta}$ and $p_j = \partial\tilde{L}/\partial\dot{q}_j$ are the canonical momenta of the ICDW ring and the environment, respectively. The fluctuating magnetic flux is given by

$$A(\mathbf{q}) = \sum_{j=1}^{\mathcal{N}} cq_j.$$

Classically, this magnetic flux will randomly change the phase and the *mechanical* angular momentum $I\dot{\theta}$ of the ICDW ring due to electromotive force. An equivalent Lagrangian obtained by a canonical transformation is

$$L(\dot{\theta},\mathbf{R},\dot{\mathbf{R}}) = \frac{1}{2}I\dot{\theta}^2 + \frac{m}{2}\sum_j^{\mathcal{N}}\left[\dot{R}_j^2(\theta) - \omega_j^2\left(R_j(\theta) - \frac{C_j\theta}{m\omega_j^2}\right)^2\right], \tag{11}$$

which is the Lagrangian of a bath of field particles R_j coupled to the phase θ by springs. $R_j(p_j,\theta) = -\frac{p_j-c\theta}{m\omega_j}$, $P_j(q_j) = m\omega_j q_j$, and $C_j = c\omega_j$. Equations (10) and (11) are precisely the kinds of Lagrangian considered by Caldeira and Leggett, so we can use the results in [10] but with slight modifications due to the periodicity of the ring.

B. Classical solution

The equation of motion of θ obtained from the Lagrangian (11) is the generalized Langevin equation [20]

$$I\ddot{\theta}(t) + 2\int_0^t d\tau\,\alpha_I(t-\tau)\theta(\tau) = \xi(t) \tag{12}$$

with the dissipation kernel $\alpha_{\mathrm{I}}(t - \tau)$, the memory function $\gamma(t - \tau)$, and the classical fluctuating force $\xi(t)$ defined by

$$\alpha_{\mathrm{I}}(t - \tau) = I\gamma(0)\delta(t - \tau) + \frac{I}{2}\frac{d}{dt}\gamma(t - \tau),$$

$$\gamma(t - \tau) = \sum_{j=1}^{\mathcal{N}} \frac{C_j^2}{Im\omega_j^2} \cos\omega_j(t - \tau),$$

$$\xi(t) = \sum_{j=1}^{\mathcal{N}} C_j \left[R_j(0)\cos\omega_j t + \frac{P_j(0)}{m\omega_j}\sin\omega_j t \right].$$

The correlation function of the classical force is given by the noise kernel

$$\langle \xi(t)\xi(\tau) \rangle_{\mathrm{env}} = \hbar\alpha_{\mathrm{R}}(t - \tau),$$

$$\alpha_{\mathrm{R}}(t - \tau) = \sum_{j}^{\mathcal{N}} \frac{C_j^2}{2m\omega_j}\coth\left(\frac{\hbar\omega_j}{2k_{\mathrm{B}}T}\right)\cos\omega_j(t - \tau),$$

where the average $\langle \cdot \rangle_{\mathrm{env}}$ is taken with respect to the environment coordinate at equilibrium. It is convenient to define the spectral density function

$$\mathcal{J}(\omega) = \frac{\pi}{2}\sum_{j}^{\mathcal{N}} \frac{C_j^2}{m_j\omega_j}\delta(\omega - \omega_j) \tag{13}$$

and assume the power-law spectrum $\mathcal{J}(\omega) = Ig_s\omega^s$ [21] with a cutoff frequency Ω and $0 < s < 2$. Then, $\alpha_{\mathrm{R}}(t - \tau)$ can be written

$$\alpha_{\mathrm{R}}(t - \tau) = \frac{Ig_s}{\pi}\int_0^{\Omega}\omega^s\coth\left(\frac{\hbar\omega}{2k_{\mathrm{B}}T}\right)\cos\omega(t - \tau)d\omega.$$

The classical solution of θ is

$$\theta(t) = G(t)\dot{\theta}(0) + \dot{G}(t)\theta(0) + \frac{1}{I}\int_0^t d\tau G(t - \tau)\xi(t) \tag{14}$$

with the fundamental solution

$$G(t) = \mathcal{L}^{-1}\left[\frac{1}{z^2 + z\hat{\gamma}(z)}\right](t),$$

where $\mathcal{L}^{-1}$ is the inverse Laplace transform. The Laplace transform of the memory function $\gamma(t)$ can be written [11]

$$\hat{\gamma}(z) = \omega_s^{2-s}z^{s-1}, \qquad \omega_s = \left(\frac{g_s}{\sin\frac{\pi s}{2}}\right)^{1/(2-s)},$$

and $G(t)$ takes the form of a generalized Mittag-Leffler function $E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\beta)}$:

$$G(t) = tE_{2-s,2}[-(\omega_s t)^{2-s}].$$

For Ohmic damping with $s = 1$ and $\hat{\gamma}(z) = g_1 \equiv 2\gamma$, we obtain

$$G(t) = \frac{1 - e^{-2\gamma t}}{2\gamma}. \tag{15}$$

$G(t)$ and $\dot{G}(t)$ are shown in Fig. 3. We note that $\dot{G}(t) \approx 1$ for t less than some damping time scale $\tau_{\mathrm{damp},s}$. In other words, the fluctuating magnetic flux does not affect the dynamics of an ICDW ring for $t < \tau_{\mathrm{damp},s}$ and $e^{i\theta(t)}$ oscillates periodically with period $P \approx 4\pi\mu$. Next, we quantize the ICDW ring + environment system to show that this oscillation is observable for a finite time τ_Q and forms an effective QTC as a metastable state.
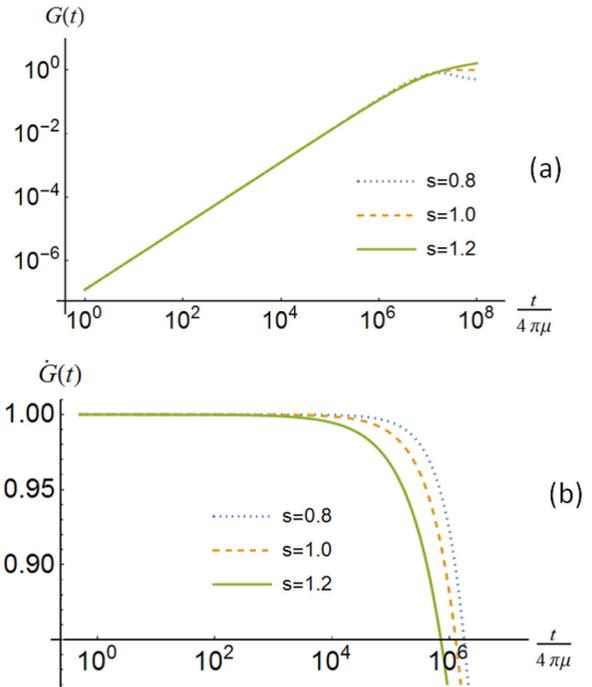

FIG. 3. These plots are shown with $g_s = 1\ \mathrm{Hz}^{2-s}$ and $\mu = 10^{-8}$ sec. (a) The fundamental solution $G(t)$ of a classical ICDW ring coupled to its environment is shown for sub-Ohmic ($s < 1$), Ohmic ($s = 1$), and super-Ohmic ($s > 1$) damping. Note that $G(t) \approx t$ for t less than some time scale $\tau_{\mathrm{damp},s}$. The time t is normalized by $4\pi\mu$, so the horizontal axis gives the number of "lattice points." (b) The derivative of the fundamental solution, $\dot{G}(t)$, heuristically describes the velocity of the ICDW ring. Note that $\dot{G}(t) \approx 1$ for $t \ll \tau_{\mathrm{damp},s}$.

C. Quantization of ICDW ring coupled to environment

The ICDW ring + environment system is quantized using the commutation relations $[\hat{\tilde{\pi}}_\theta, \hat{W}] = \hbar\hat{W}$ and $[\hat{q}_j, \hat{p}_k] = i\hbar\delta_{jk}$. Define the orthonormal position state $|\mathbf{q}\rangle = \prod_{i=1}^{\mathcal{N}}|q_i\rangle$ and the orthonormal momentum state $|\mathbf{p}\rangle = \prod_{i=1}^{\mathcal{N}}|p_i\rangle$ such that

$$\langle\mathbf{q}|\hat{q}_j|\psi\rangle = q_j\langle\mathbf{q}|\psi\rangle, \qquad \langle\mathbf{q}|\hat{p}_j|\psi\rangle = -i\hbar\frac{\partial}{\partial q_j}\langle\mathbf{q}|\psi\rangle, \tag{16}$$

and the inner product of $|\mathbf{q}\rangle$ and $|\mathbf{p}\rangle$ is defined as $\langle\mathbf{q}|\mathbf{p}\rangle = \frac{1}{\sqrt{2\pi\hbar}^{\mathcal{N}}}\exp\left(\frac{i}{\hbar}\mathbf{q}\cdot\mathbf{p}\right)$. The periodic boundary condition of the ring implies that $\langle\theta + 2\pi n|\tilde{\pi}_\theta\rangle = \langle\theta|\tilde{\pi}_\theta\rangle$ for some integer n; hence the angular momentum eigenstates are quantized $\langle\psi_l|\hat{\tilde{\pi}}_\theta|\psi_l\rangle = l\hbar, l = 0, \pm, \pm 2, \dots, \qquad |\tilde{\pi}_\theta\rangle = \hbar^{-1/2}|\psi_l\rangle,$ $\langle\theta|\psi_l\rangle = \frac{1}{\sqrt{2\pi}}e^{il\theta}$. Moreover, one can easily show that

$$\langle\theta,\mathbf{p}|\tilde{\pi}_\theta,\mathbf{q}\rangle = \langle\theta,\mathbf{R}(\theta)|\pi_\theta,\mathbf{P}\rangle. \tag{17}$$

$\hat{W}$ is independent of the environmental coordinate. So, the expectation value of $\hat{W}$ is

$$\langle\hat{W}(t)\rangle = \frac{1}{2}\int_{-\pi}^{\pi}d\theta_{\mathrm{f}}\int_{-\pi}^{\pi}d\phi_{\mathrm{f}}\rho(\theta_{\mathrm{f}},\phi_{\mathrm{f}},t)\langle\phi_{\mathrm{f}}|\hat{W}|\theta_{\mathrm{f}}\rangle$$
$$+ \frac{1}{2}\int_{-\pi}^{\pi}d\theta_{\mathrm{f}}\int_{-\pi}^{\pi}d\phi_{\mathrm{f}}\langle\theta_{\mathrm{f}}|\hat{W}|\phi_{\mathrm{f}}\rangle\rho(\phi_{\mathrm{f}},\theta_{\mathrm{f}},t),$$

where the reduced density matrix of the ICDW ring is (see Appendix A)

$$\rho(\theta_f, \phi_f, t) = \int_{-\pi}^{\pi} d\theta_i \int_{-\pi}^{\pi} d\phi_i \sum_{l_1, l_2 \in \mathbb{Z}} \rho(\theta_i, \phi_i, 0)$$

$$\times J(\theta_f + 2\pi l_1, \phi_f + 2\pi l_2, t; \theta_i, \phi_i, 0). \quad (18)$$

The exact form of $J(\theta_f, \phi_f, t; \theta_i, \phi_i, 0)$ for Ohmic dissipation $s = 1$ was calculated in [10]. For general damping with arbitrary s the computation of the reduced density matrix is essentially equivalent to [10] and we obtain

$$J(\theta_f, \phi_f, t; \theta_i, \phi_i, 0) = F^2(t) \exp\left(\frac{i}{\hbar} S[\varphi_{cl}^+, \varphi_{cl}^-] - \Gamma[\varphi_{cl}^-]\right).$$

$$(19)$$

$\varphi_{cl}^+ = \frac{1}{2}(\theta_{cl} + \phi_{cl})$ and $\varphi_{cl}^- = \phi_{cl} - \theta_{cl}$ are the classical coordinates obtained from the Euler-Lagrange equation:

$$I\ddot{\varphi}_{cl}^-(u) + 2 \int_u^t d\tau \varphi_{cl}^-(\tau) \alpha_I(\tau - u) = 0, \quad (20)$$

$$I\ddot{\varphi}_{cl}^+(u) + 2 \int_0^u d\tau \varphi_{cl}^+(\tau) \alpha_I(u - \tau) = 0, \quad (21)$$

whose solution is given in terms of boundary conditions $\varphi_i^{\pm} = \varphi_{cl}^{\pm}(0), \varphi_f^{\pm} = \varphi_{cl}^{\pm}(t)$:

$$\varphi_{cl}^+(u) = \kappa_i(u; t)\varphi_i^+ + \kappa_f(u; t)\varphi_f^+, \quad (22)$$

$$\varphi_{cl}^-(u) = \kappa_i(t - u; t)\varphi_f^- + \kappa_f(t - u; t)\varphi_i^-, \quad (23)$$

$$\kappa_i(u; t) = \dot{G}(u) - \frac{\dot{G}(t)}{G(t)} G(u), \quad \kappa_f(u; t) = \frac{G(u)}{G(t)}. \quad (24)$$

The classical action and the noise action are given by

$$S[\varphi_{cl}^+, \varphi_{cl}^-] = S_{cl}(\varphi_f^+, \varphi_f^-, t; \varphi_i^+, \varphi_i^-, 0)$$

$$= -I[\dot{\varphi}_{cl}^+(t)\varphi_f^- - \dot{\varphi}_{cl}^+(0)\varphi_i^-],$$

$$\Gamma[\varphi_{cl}^-] = \Gamma_{cl}(\varphi_f^-, t; \varphi_i^-, 0)$$

$$= \frac{1}{2\hbar} \int_0^t d\tau \int_0^t d\tau' \varphi^-(\tau) \alpha_R(\tau - \tau')\varphi^-(s).$$

$F^2(t)$ is a normalization function such that $\text{tr}[\rho(t)] = \langle 1 \rangle = 1$. The winding numbers l_1 and l_2 can be absorbed into the θ_f and ϕ_f integrals, respectively, by changing the domain of θ_f and ϕ_f from S^1 to $\mathbb{R}^1$. Then, taking care of the non-Hermiticity of $\hat{W}$, we obtain

$$\langle \hat{W}(t) \rangle = \frac{r_1^+(t) + r_1^-(t)}{r_2^+(t) + r_2^-(t)}, \quad (25)$$

$$r_1^+ = \int_{-\pi}^{\pi} d\theta_i \sum_{n \in \mathbb{S}_1(\theta, t)} \rho(\theta_i - f_1(t), \theta_i, 0)$$

$$\times e^{-in\pi - i\mu\theta \dot{f}_1(t) + \frac{i\mu}{2} f_1(t)\dot{f}_1(t) - \Gamma_{cl}(2\pi n, t; f_1(t), 0)},$$

$$r_1^- = \int_{-\pi}^{\pi} d\theta_i \sum_{n \in \mathbb{S}_1(\theta, t)} \rho(\theta_i, \theta_i + f_1(t), 0)$$

$$\times e^{-in\pi - i\mu\theta \dot{f}_1(t) - \frac{i\mu}{2} f_1(t)\dot{f}_1(t) - \Gamma_{cl}(2\pi n, t; f_1(t), 0)},$$

$$r_2^+ = \int_{-\pi}^{\pi} d\theta_i \sum_{n \in \mathbb{S}_2(\theta, t)} \rho(\theta_i - f_2(t), \theta_i, 0)$$

$$\times e^{-i\mu\theta \dot{f}_2(t) + \frac{i\mu}{2} f_2(t)\dot{f}_2(t) - \Gamma_{cl}(2\pi n, t; f_2(t), 0)},$$

$$r_2^- = \int_{-\pi}^{\pi} d\theta_i \sum_{n \in \mathbb{S}_2(\theta, t)} \rho(\theta_i, \theta_i + f_2(t), 0)$$

$$\times e^{-i\mu\theta \dot{f}_2(t) - \frac{i\mu}{2} f_2(t)\dot{f}_2(t) - \Gamma_{cl}(2\pi n, t; f_2(t), 0)},$$

with $f_1(t) = 2\pi n\dot{G}(t) - G(t)/\mu$, $f_2(t) = 2\pi n\dot{G}(t)$, $\mathbb{S}_1(\theta, t) = \{n \in \mathbb{Z} | -\pi < \theta + f_1(t) < \pi\}$, and $\mathbb{S}_2(\theta, t) = \{n \in \mathbb{Z} | -\pi < \theta + f_2(t) < \pi\}$. This is the most general form of the expectation value of $\hat{W}$ for an ICDW ring coupled to its environment. Although the derivation of (25) is exact, it is not very insightful, so we make some approximations.

D. Early-time approximation

Let us consider the classical solution (22) with $\varphi_{cl}^+ = \theta_f$ and Ohmic damping (15); then we see immediately that $\dot{\theta}(t) \sim \dot{\theta}(0)e^{-2\gamma t}$. Therefore, we are interested in the range $t \ll 1/(2\gamma) \equiv \tau_{damp,1}$. For general dissipation, we can see from Fig. 3 that there exists a time scale $\tau_{damp,s}$ such that $G(t) \approx t$, $\dot{G}(t) \approx 1$ for $t \ll \tau_{damp,s}$. Then, writing $t = 2\pi I(m + a)/\hbar$ for an integer m and $0 < a < 1$, we have $\mathbb{S}_1(\theta < -2a\pi, t) = \{m + 1\}$, $\mathbb{S}_1(\theta > -2a\pi, t) = \{m\}$, and $\mathbb{S}_2(\theta, t) = \{0\}$. Therefore, we conclude that $\mathbb{S}_1(\theta, t)$ is approximately the $(m/2)$th lattice point. Then, using $\dot{f}_1(t) \approx -\dot{G}(t)\hbar/I$ we obtain the approximate form

$$\langle \hat{W}(t) \rangle \approx \int_{-\pi}^{\pi} d\theta_i \sum_{n \in \mathbb{S}_1(\theta, t)} \rho(\theta_i, \theta_i - G(t)/\mu, 0)$$

$$\times \exp\left\{ i\theta_i \dot{G}(t) - i\frac{G(t)\dot{G}(t)}{2\mu} - \Gamma_{cl}(2\pi n, t; f_1(t), 0) \right\}$$

$$+ \int_{-\pi}^{\pi} d\theta_i \sum_{n \in \mathbb{S}_1(\theta, t)} \rho(\theta_i + G(t)/\mu, \theta_i, 0)$$

$$\times \exp\left\{ i\theta_i \dot{G}(t) + i\frac{G(t)\dot{G}(t)}{2\mu} - \Gamma_{cl}(2\pi n, t; f_1(t), 0) \right\}.$$

For $t \ll \tau_{damp,s}$ we have $\varphi_{cl}^-(\tau) \approx \tau/\mu$ and the noise action can be written

$$\Gamma_{cl}(2\pi n, t; f_1(t), 0) \approx \Gamma_{T,s}(t)$$

$$= \frac{g_s}{2\pi\mu} \int_0^{\Omega} d\omega \coth\left(\frac{\hbar\omega}{2k_B T}\right) \Upsilon(\omega),$$

$$\Upsilon(\omega) = \omega^{s-4}(2 + \omega^2 t^2 - 2\cos\omega t - 2\omega t \sin\omega t).$$

Taking the low-temperature limit $\frac{\hbar\Omega}{2k_B T} \to \infty$ such that $\coth\frac{\hbar\Omega}{2k_B T} \to 1$ we obtain

$$\Gamma_{T,s}(t) = -\frac{g_s}{\pi\mu} \frac{\Omega^{s-3}\left({}_1F_2\left(\frac{s-3}{2}; \frac{1}{2}, \frac{s-1}{2}; -\frac{1}{4}t^2\Omega^2\right) - 1\right)}{s - 3}$$

$$- \frac{g_s t^2}{2\pi\mu} \frac{\Omega^{s-1}\left({}_1F_2\left(\frac{s-1}{2}; \frac{1}{2}, \frac{s+1}{2}; -\frac{1}{4}t^2\Omega^2\right) - 1\right)}{s - 1},$$

where ${}_1F_2(a_1; b_1, b_2; z)$ is the generalized hypergeometric function. Define the decoherence time $\tau_{decoh,s}$ such that

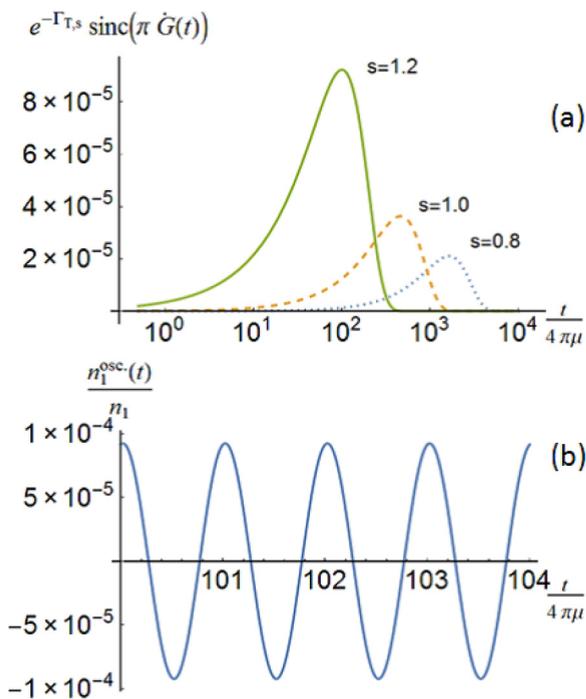FIG. 4. (a) The amplitude of the charge density oscillation is shown for $g_s = 1$ Hz^{2-s}, $\mu = 10^{-8}$ sec, and $\Omega = 1/\mu$. This oscillation is an effective QTC. In general, the oscillation amplitude is larger for super-Ohmic ($s > 1$) damping but has a small lifetime τ_Q. On the other hand, the oscillation amplitude is small for sub-Ohmic ($s < 1$) damping but has a long τ_Q. (b) The charge density oscillation is shown for super-Ohmic damping with $s = 1.2$, $g_{1.2} = 1$ Hz$^{0.8}$, $\mu = 10^{-8}$ sec, and $\Omega = 1/\mu$. The time t is normalized such that the horizontal axis gives the number of "lattice points."

$\Gamma_{T,s}(\tau_{\text{decoh},s}) = 1$. Numerical analysis shows that the order of $\tau_{\text{decoh},s}$ does not change with $\Omega < 1/(2\mu)$ and decreases rapidly for $\Omega > 1/(2\mu)$. If we set $\Omega \sim 1/\mu$ and use the ground state ρ_0, then for $t \ll \tau_Q \equiv \min\{\tau_{\text{damp},s}, \tau_{\text{decoh},s}\}$ we have

$$n(x,t) \approx n_0 + n_1^{\text{osc.}}(t) \cos(2k_F x), \qquad (26)$$

$$n_1^{\text{osc.}}(t) = n_1 \text{sinc}[\pi \dot{G}(t)] \cos\left[\frac{\dot{G}(t)G(t)}{2\mu}\right] e^{-\Gamma_{T,s}(t)}. \qquad (27)$$

Equation (26) is the main result of this paper. It shows that the amplitude of an ICDW ring threaded by a (time-independent) fluctuating magnetic flux oscillates for a finite time τ_Q and forms an effective QTC. In the no-damping limit $\hat{\gamma}(z) \to 0$ we have $G(t) \to t, \dot{G}(t) \to 1$ and recover $\langle \hat{W}(t) \rangle = 0$. This charge density oscillation is shown in Fig. 4.

IV. DISCUSSION

First, we elaborate our assumption of ICDW ring. A mathematical definition of ICDW is that λ/a is an irrational number. We note that a CDW formed on a macroscopic crystal is basically incommensurate because the wavelength of a CDW is given by $\lambda = \pi/k_F$, where the Fermi wave number k_F is usually an irrational number for an arbitrary band filling. However, strictly speaking, λ/a of a finite-size system can never be irrational. A physical condition is that

the commensurability pinning energy is negligible, which is possible if λ/a cannot be expressed as a simple fraction like 2, 5/2, etc. In order to explain this, suppose that for some integer $M \geqslant 2$, Ma/λ is an integer. In other words, the same atom-electron configuration is obtained if we move the CDW by M wavelengths and $\epsilon_{k+2Mk_F} = \epsilon_k$ (ϵ_k is the energy of an electron with momentum $\hbar k$). Then, the energy required to move a CDW by a small phase ϕ from its equilibrium is [22]

$$\epsilon(\phi; M) = \frac{|\Delta|^2}{\epsilon_F} \left(\frac{e|\Delta|}{W}\right)^{M-2} \frac{M\phi^2}{2},$$

where ϵ_F is the Fermi energy, W is the bandwidth, $|\Delta|$ is the CDW gap width, and e is the elementary charge in natural units. We see that $\epsilon(\phi; M)$ approaches zero rapidly for large M as the distinction between rational and irrational numbers becomes academic. For example, for a ring crystal with N_a atoms and N_λ CDW wavelengths, we obtain $\lambda/a = N_a/N_\lambda$. Therefore, for a large N_a and a large N_λ, M can always be arbitrarily large (order of N_a); hence the commensurability energy is completely negligible. In fact, we can experimentally make submicrometer-scale ICDW rings such that, we expect, N_a and N_λ are not so large but M is very large such that commensurability energy is negligible.

Usually, superposition of ICDWs with different phases is not observed because of impurity pinning. However, the probability of impurity decreases with decreasing radius. Commensurability effect may become significant for small radius (more precisely, for some small M). But, the origin of the time crystal periodicity in our model is the uncertainty principle, which appears as a collective fluctuation of the ICDW phase. If the commensurability effect becomes significant, then the ICDW phase is expected to fluctuate periodically around some phase $\theta = \theta_0$ determined by the ion's position. This fluctuation is expected to become apparent and oscillate periodically by coupling to environment (fluctuating magnetic flux).

Next, we discuss the presence of an upper bound and a lower bound for the radius of the ICDW ring in our model. Decoherence-induced breaking of time translation symmetry occurs, in principle, only in mesoscopic systems: There is an upper bound for the CDW radius determined by the coupling strength $\gamma = \omega_s/2$ and a lower bound given by the CDW wavelength λ. We will first calculate the upper bound by replacing the environment with an equivalent LC circuit. Here, we focus on Ohmic damping because an approximate form of γ can be calculated explicitly. In general, the coupling strength γ depends on the CDW radius R. In order to explicitly see the R dependence, suppose that the fluctuating magnetic flux in our model comes from an external coil connected to a series of parallel capacitors. Then, the Lagrangian of the CDW + environment system is

$$L_{\text{circuit}} = L_0 + MI_{CDW} \sum_{j=1}^{\mathcal{N}} I_j + \sum_{j=1}^{\mathcal{N}} \frac{m}{2}(I_j^2 - \omega_j^2 Q_j^2).$$

L_0 is the Lagrangian of an isolated ICDW ring, $I_{CDW} = e\dot{\theta}/\pi$ is the CDW current induced by the fluctuating flux, Q_j is the net charge on the capacitor j with capacitance $\mathscr{C}_j$, m is the inductance of the coil, $\omega_j = 1/\sqrt{m\mathscr{C}_j}$, and M is the mutual inductance between the coil and the CDW. We

immediately see that L_{circuit} is equivalent to the Lagrangian in Eq. (10) but with the interaction Lagrangian replaced by $\dot{\theta} \sum_{j=1}^{\mathcal{N}} \frac{Me}{\pi} \dot{Q}$. Therefore, define $p_j = mI_j$, $C_j = Me\omega_j^2/\pi$ and we obtain the Lagrangian in Eq. (11) after the canonical transformation

$$L = L_{\text{circuit}} - \frac{d}{dt} \sum_j^{\mathcal{N}} (MI_{CDW} + p_j)Q_j$$

with $P_j = m\dot{R}_j = m\omega_j Q_j$ and $R_j = -(p_j - MI_{CDW})/(m\omega_j)$. Assume that the radius of the coil r_{coil} is much larger than the CDW radius R and that the coil and the CDW are concentric. Then, we obtain $M = (\mu_0\pi R^2)/(2r_{\text{coil}})$. Next, integrate (13) with respect to ω from $\omega = 0$ to $\omega = \Omega = 1/\mu$ and obtain $\gamma = \beta R$, $\beta = (\pi\mu_0^2 e^2 \rho c_0^6)/(32\hbar r_{\text{coil}}^2 mv_F^3)$. Here, μ_0 is the permeability of free space and ρ is the density of states defined by $\sum_j \rightarrow \int d\omega\rho$. If we assume that $\rho \sim \Omega$ like in the Caldeira-Leggett model, then the order of β may change depending on the parameters of the CDW and the parameters of the coil, but $\gamma = \beta R$ is usually smaller than 1 Hz. Now, if the radius of the CDW ring is too large, then periodicity does not appear because the oscillation period exceeds the lifetime τ_Q of the time crystal. The upper bound of the CDW radius R is given by the condition $N > 1$, where $N = \min\{\tau_{\text{damp},s}, \tau_{\text{deph},s}\}/(4\pi\mu)$ is the number of oscillations. For Ohmic damping ($s = 1$) we have the approximate form $\tau_{\text{damp},1} = \gamma^{-1}$ and $\tau_{\text{deph},1} = \sqrt{\mu\gamma^{-1}} = \sqrt{\mu\gamma}\tau_{\text{damp},1}$. Let $\mu\gamma < 1$, i.e., $\tau_{\text{damp},1} > \tau_{\text{deph},1}$ (which is a valid assumption because a typical value for μ with radius 10^{-6} m is 10^{-6} s and γ is usually smaller than 1 Hz); then, the upper bound to observe more than one oscillation is $R < \frac{c_0}{4\pi\sqrt{v_F\beta}}$ which is typically 1 mm. There is also a lower bound determined by the CDW wavelength λ: The radius should be large enough to define a Fermi surface. This condition is given by $p_F = \hbar\pi/\lambda \gg \hbar/R$. In other words, R should be much larger than λ.

Finally, we discuss how our model can be tested experimentally and discuss how our results may be applicable to other physical systems. Ring-shaped crystals and ICDW ring crystals (such as monoclinic TaS$_3$ ring crystals and NbSe$_3$ ring crystals) have been produced and studied by the Hokkaido group [7–9]. Therefore, our model can be tested provided ring crystals with almost no defects and impurities can be produced. The oscillation in (26) implies that the local charge density of the ICDW ring oscillates with frequency $\omega = \hbar/2I$. For a ring with diameter $2R = 1$ μm, $v_F/c_0 = 10^3$, and $v_F = 10^5$ m/s, we have $\omega = 10^8$ Hz. The time dependence of the charge density modulation can be measured using scanning tunneling microscopy (STM) [23] and/or using narrow-band noise with vanishing threshold voltage [24].

We recall that the origin of the quantum oscillation in our model is the uncertainty principle. If we were to consider a single particle with mass m^* confined on a ring with radius R, then the particle's wave packet expands with a velocity $v \sim \hbar/(4m^*R)$ and $e^{i\theta}$ oscillates with period $P = 2\pi R/v$. However, a charge density wave has an internal periodicity given by the wavelength λ. Then the period of oscillation is $P \sim \lambda/v$. Therefore, because an ICDW ring is described

by a macroscopic wave function with internal periodicity, the number of lattice points N in our model is numerous. And yet, the periodicity of $\hat{W}(t)$ seems to be universal for any wave functions on S^1 (ring system). Therefore, our results may be applicable to earlier models of QTC such as [1,2,25] and annular Josephson junctions [26]. Moreover, it was shown very recently in [27] that the ground state of a ^{40}Ca$^+$ ring trap possesses rotational symmetry as the number of ions is decreased. Our results predict that quantum oscillations may appear in such ring traps with the appropriate setup. We also recall that Volovik's proposal of metastable effective QTC [3] is not restricted to ring systems. Therefore, time translation symmetry breaking by *decoherence* may occur in other systems coupled to time-independent environment, and without a periodic driving field. We also expect that many other incommensurate systems such as incommensurate spin density waves [5], incommensurate mass density waves [28–30], or possibly some dielectrics that exhibit incommensurate phases [31] may be used to test our results and to model QTC without spontaneous symmetry breaking.

APPENDIX: QUANTUM BROWNIAN MOTION ON S^1

1. Reduced density matrix on S^1

Consider an ICDW ring A coupled to its environment B. Let ρ_A and ρ_B denote the density operators of A and B, respectively. Let θ and ϕ be angular coordinates on the ring system A and let $\mathbf{p} = \{p_k : k = 1, \mathcal{N}\}$ and $\mathbf{s} = \{s_k : k = 1, \mathcal{N}\}$ be momentum coordinates of the bath B. Then, the density matrix of the coupled system can be written

$$\rho(\theta,\mathbf{p},\phi,\mathbf{s}) = \langle\theta,\mathbf{p}|\rho_{AB}(t)|\phi,\mathbf{s}\rangle$$
$$= \int_0^{2\pi} d\theta' d\phi' \int_{-\infty}^{\infty} d\mathbf{p}' d\mathbf{s}' K(\theta,\mathbf{p},t;\theta',\mathbf{p}',0)$$
$$\times K^*(\phi,\mathbf{s},t;\phi',\mathbf{s}',0)\langle\theta',\mathbf{p}'|\rho_{AB}(0)|\phi',\mathbf{s}'\rangle,$$

where

$$K(\theta,\mathbf{p},t;\theta',\mathbf{p}',0) = \langle\theta,\mathbf{p}|e^{-\frac{i}{\hbar}\hat{H}t}|\theta',\mathbf{p}'\rangle \quad \text{(A1)}$$

and

$$K^*(\phi,\mathbf{s},t;\phi',\mathbf{s}',0) = \langle\phi',\mathbf{s}'|e^{\frac{i}{\hbar}\hat{H}t}|\phi,\mathbf{s}\rangle \quad \text{(A2)}$$

are recognized as Feynman propagators if we notice that $|\mathbf{p}\rangle$ and $|\mathbf{s}\rangle$ are actually position states after canonical transformation. The propagators (A1) and (A2) can be written using path integrals by dividing the time t into N time steps of length $\epsilon = t/(N + 1)$, $\mathbf{p} = \mathbf{p}_N$, $\mathbf{p}' = \mathbf{p}_0$, $\theta = \theta_N$, and $\theta' = \theta_0$.

For $N \to \infty$ the propagator becomes

$$K(\theta,\mathbf{p},t;\theta',\mathbf{p}',0) = \left\langle \theta,\mathbf{p} \left| \lim_{\epsilon \to 0} \left(\exp -\frac{i\epsilon}{\hbar} \hat{H} \right)^N \right| \theta',\mathbf{p}' \right\rangle = \lim_{\epsilon \to 0} \int_0^{2\pi} \left(\prod_{n=1}^{N-1} d\theta_n \right) \int_{-\infty}^{\infty} \left(\prod_{n=1}^{N-1} d\mathbf{p}_n \right) \prod_{n=1}^{N}$$

$$\times K_n(\theta_n,\mathbf{p}_n,\epsilon;\theta_{n-1},\mathbf{p}_{n-1},0).$$

The Hamiltonian operator $\hat{H}$ can be decomposed into the kinetic part $\hat{\mathcal{K}}$ and the potential part $\hat{\mathcal{V}}$, i.e., $\hat{H} = \hat{\mathcal{K}} + \hat{\mathcal{V}}$, which satisfy the eigenvalue equations

$$\hat{\mathcal{K}} |\psi_l,\mathbf{q}\rangle = \mathcal{K}(l,\mathbf{P}) |\psi_l,\mathbf{q}\rangle, \quad \hat{\mathcal{V}} |\theta,\mathbf{p}\rangle = \mathcal{V}(\theta,\mathbf{R}(\theta)) |\theta,\mathbf{p}\rangle. \tag{A3}$$

Then we obtain

$$K(\theta_n,\mathbf{p}_n,\epsilon;\theta_{n-1},\mathbf{p}_{n-1},0) = \sum_{l_n} \int_{-\infty}^{\infty} d\mathbf{q}_n \left\langle \theta_n,\mathbf{p}_n \left| \exp\left(-\frac{i\epsilon}{\hbar} \hat{\mathcal{K}} \right) \right| \psi_{l_n},\mathbf{q}_n \right\rangle \left\langle \psi_{l_n},\mathbf{q}_n \left| \exp\left(-\frac{i\epsilon}{\hbar} \hat{\mathcal{V}} \right) \right| \theta_{n-1},\mathbf{p}_{n-1} \right\rangle$$

$$= \sum_{l_n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\mathbf{q}_n}{(2\pi\hbar)^{\mathcal{N}}} \exp\left\{ -\frac{i\epsilon}{\hbar} \mathcal{K}(l_n,\mathbf{P}_n) - \frac{i\epsilon}{\hbar} \mathcal{V}(\theta_{n-1},\mathbf{R}_{n-1}(\theta_{n-1})) \right\}$$

$$\times \exp\left\{ il_n(\theta_n - \theta_{n-1}) - \frac{i}{\hbar} \mathbf{q}_n \cdot (\mathbf{p}_n - \mathbf{p}_{n-1}) \right\}.$$

Next, define $A_n = \sum_j c q_{j,n} = \sum_j \frac{C_j}{m\omega_j^2} P_{j,n}$ and obtain

$$K(\theta_n,\mathbf{p}_n,\epsilon;\theta_{n-1},\mathbf{p}_{n-1},0) = \sum_{l_n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\mathbf{q}_n}{(2\pi\hbar)^{\mathcal{N}}} \exp\left\{ -\frac{i\epsilon}{\hbar} \mathcal{K}(l_n,\mathbf{P}_n) - \frac{i\epsilon}{\hbar} \mathcal{V}(\theta_{n-1},\mathbf{R}_{n-1}(\theta_{n-1})) \right\}$$

$$\times \exp\left\{ i(l_n - A_n/\hbar)(\theta_n - \theta_{n-1}) + \frac{i}{\hbar} \mathbf{P}_n \cdot (\mathbf{R}_n(\theta_n) - \mathbf{R}_{n-1}(\theta_{n-1})) \right\}.$$

The sum over l_n can be replaced by a sum of integrals using the Poisson resummation formula

$$\sum_{l \in \mathbb{Z}} f(l) = \sum_{l \in \mathbb{Z}} \int_{-\infty}^{\infty} f(\zeta) \exp(2\pi l \zeta) d\zeta. \tag{A4}$$

Then,

$$K(\theta_n,\mathbf{p}_n,\epsilon;\theta_{n-1},\mathbf{p}_{n-1},0) = \sum_{l_n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\zeta_n}{2\pi} \int_{-\infty}^{\infty} \frac{d\mathbf{q}_n}{(2\pi\hbar)^{\mathcal{N}}} \exp\left\{ -\frac{i\epsilon}{\hbar} \mathcal{K}(\zeta_n,\mathbf{P}_n) - \frac{i\epsilon}{\hbar} \mathcal{V}(\theta_{n-1},\mathbf{R}_{n-1}(\theta_{n-1})) \right\}$$

$$\times \exp\left\{ i(\zeta_n - A_n/\hbar)(\theta_n - \theta_{n-1} + 2\pi l_n) + \frac{i}{\hbar} \mathbf{P}_n \cdot (\mathbf{R}_n(\theta_n + 2\pi l_n) - \mathbf{R}_{n-1}(\theta_{n-1})) \right\}.$$

For our Hamiltonian of an ICDW ring threaded by a fluctuating magnetic flux we have

$$\mathcal{K}(\zeta_n,\mathbf{P}_n) = \frac{(\zeta_n \hbar - A_n)^2}{2I} + \sum_j \frac{1}{2m} P_{j,n}^2, \mathcal{V}(\theta_{n-1},\mathbf{R}_{n-1}(\theta_{n-1})) = \sum_j \frac{1}{2} m\omega_j^2 \left(R_{j,n-1}(\theta_{n-1}) - \frac{C_j \theta_{n-1}}{m\omega_j^2} \right)^2.$$

We note that that the potential $\mathcal{V}(\theta,\mathbf{R}(\theta))$ is rotationally invariant. That is, for an arbitrary rotation $\theta \to \theta + \delta$, the potential is $\mathcal{V}(\theta + \delta,\mathbf{R}(\theta + \delta)) = \mathcal{V}(\theta,\mathbf{R}(\theta))$. Let $\tilde{\zeta}_n = \zeta_n - A_n/\hbar$. We use the fact that the phase space volume is conserved under canonical transformation. Then,

$$K(\theta,\mathbf{p},t;\theta',\mathbf{p}',0) = \left(\prod_{j=1}^{\mathcal{N}} \frac{1}{m\omega_j} \right) \lim_{\epsilon \to 0} \left(\prod_{n=1}^{N-1} \int_0^{2\pi} d\theta_n \int_{-\infty}^{\infty} d\mathbf{R}_n(\theta_n) \right) \left(\prod_{n=1}^{N} \sum_{l_n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\mathbf{P}_n}{(2\pi\hbar)^{\mathcal{N}}} \frac{d\tilde{\zeta}_n}{2\pi} \right)$$

$$\times \exp \sum_{n=1}^{N} \left\{ -\frac{i\epsilon}{\hbar} \left(\frac{\hbar^2}{2I} \tilde{\zeta}_n^2 + \sum_j \frac{1}{2m} P_{j,n}^2 \right) - \frac{i\epsilon}{\hbar} \mathcal{V}(\theta_{n-1},\mathbf{R}_{n-1}(\theta_{n-1})) \right\}$$

$$\times \exp \sum_{n=1}^{N} \left\{ i\tilde{\zeta}_n(\theta_n - \theta_{n-1} + 2\pi l_n) + \frac{i}{\hbar} \mathbf{P}_n \cdot [\mathbf{R}_n(\theta_n + 2\pi l_n) - \mathbf{R}_{n-1}(\theta_{n-1})] \right\}.$$

Solve the $\tilde{\zeta}_n$ and $P_{j,n}$ integrals to obtain

$$K(\theta,\mathbf{p},t;\theta',\mathbf{p}',0) = \left(\prod_{j=1}^{\mathcal{N}} \frac{1}{m\omega_j}\right) \lim_{\epsilon\to 0} \left(\prod_{n=1}^{N-1} \int_0^{2\pi} d\theta_n \int_{-\infty}^{\infty} d\mathbf{R}_n(\theta_n)\right) \left(\prod_{n=1}^{N} \sum_{l_n=-\infty}^{\infty} \sqrt{\frac{I}{2\pi i\epsilon\hbar}} \sqrt{\frac{m}{2\pi i\epsilon\hbar}}^{\mathcal{N}}\right)$$

$$\times \exp\sum_{n=1}^{N} \left\{\frac{i}{\hbar}\frac{I}{2\epsilon}(\theta_n - \theta_{n-1} + 2\pi l_n)^2 + \frac{i}{\hbar}\frac{m}{2\epsilon}[\mathbf{R}_n(\theta_n + 2\pi l_n) - \mathbf{R}_{n-1}(\theta_{n-1})]^2\right\}$$

$$\times \exp\sum_{n=1}^{N} \left\{-\frac{i\epsilon}{\hbar}\mathcal{V}(\theta_{n-1},\mathbf{R}_{n-1}(\theta_{n-1}))\right\}.$$

The integral including θ_{N-1} is

$$I_{N-1} = \sum_{l_N=-\infty}^{\infty} \sum_{l_{N-1}=-\infty}^{\infty} \sqrt{\frac{I}{2\pi i\epsilon\hbar}} \sqrt{\frac{m}{2\pi i\epsilon\hbar}}^{\mathcal{N}} \int_0^{2\pi} d\theta_{N-1} \int_{-\infty}^{\infty} d\mathbf{R}_{N-1}(\theta_{N-1})$$

$$\times \exp\left\{\frac{i}{\hbar}\frac{I}{2\epsilon}(\theta_N - \theta_{N-1} + 2\pi l_N)^2 + \frac{i}{\hbar}\frac{I}{2\epsilon}(\theta_{N-1} - \theta_{N-2} + 2\pi l_{N-1})^2\right\} \exp\left\{\frac{i}{\hbar}\frac{m}{2\epsilon}[\mathbf{R}_N(\theta_N + 2\pi l_N) - \mathbf{R}_{N-1}(\theta_{N-1})]^2\right\}$$

$$\times \exp\left\{\frac{i}{\hbar}\frac{m}{2\epsilon}[\mathbf{R}_{N-1}(\theta_{N-1} + 2\pi l_{N-1}) - \mathbf{R}_{N-2}(\theta_{N-2})]^2\right\} \exp\left\{-\frac{i\epsilon}{\hbar}\mathcal{V}(\theta_{N-1},\mathbf{R}_{N-1}(\theta_{N-1}))\right\}.$$

The sum over l_{N-1} can be absorbed into the θ_{N-1} integral by changing the domain of θ_{N-1} integration from $[0,2\pi)$ to $(-\infty,\infty)$. This is done by the following procedure:

(1) Transform l_N to $\tilde{l}_N = l_N + l_{N-1}$ and write $\tilde{\theta}_{N-1} = \theta_{N-1} + 2\pi l_{N-1}$ to obtain

$$I_{N-1} = \sum_{l_N=-\infty}^{\infty} \sum_{l_{N-1}=-\infty}^{\infty} \sqrt{\frac{I}{2\pi i\epsilon\hbar}} \sqrt{\frac{m}{2\pi i\epsilon\hbar}}^{\mathcal{N}} \int_0^{2\pi} d\theta_{N-1} \int_{-\infty}^{\infty} d\mathbf{R}_{N-1}(\theta_{N-1})$$

$$\times \exp\left\{\frac{i}{\hbar}\frac{I}{2\epsilon}(\tilde{\theta}_N - \theta_{N-1} + 2\pi\tilde{l}_N)^2 + \frac{i}{\hbar}\frac{I}{2\epsilon}(\tilde{\theta}_{N-1} - \theta_{N-2})^2\right\} \exp\left\{\frac{i}{\hbar}\frac{m}{2\epsilon}[\mathbf{R}_N(\theta_N + 2\pi\tilde{l}_N) - \mathbf{R}_{N-1}(\tilde{\theta}_{N-1})]^2\right\}$$

$$\times \exp\left\{\frac{i}{\hbar}\frac{m}{2\epsilon}[\mathbf{R}_{N-1}(\tilde{\theta}_{N-1}) - \mathbf{R}_{N-2}(\theta_{N-2})]^2\right\} \exp\left\{-\frac{i\epsilon}{\hbar}\mathcal{V}(\tilde{\theta}_{N-1},\mathbf{R}_{N-1}(\tilde{\theta}_{N-1}))\right\}.$$

(2) Change the variable of integration $\int_0^{2\pi} d\theta_n \to \int_{2\pi n}^{2\pi(n+1)} d\tilde{\theta}_n$ and change the domain of θ_{N-1} integration from $[0,2\pi)$ to $(-\infty,\infty)$ and use the periodicity of $\mathcal{V}(\theta_{N-1},\mathbf{R}_{N-1}(\theta_{N-1}))$ to obtain

$$I_{N-1} = \sum_{l_N=-\infty}^{\infty} \sqrt{\frac{I}{2\pi i\epsilon\hbar}} \sqrt{\frac{m}{2\pi i\epsilon\hbar}}^{\mathcal{N}} \int_{-\infty}^{\infty} d\theta_{N-1} \int_{-\infty}^{\infty} d\mathbf{R}_{N-1}(\theta_{N-1})$$

$$\times \exp\left\{\frac{i}{\hbar}\frac{I}{2\epsilon}(\theta_N - \theta_{N-1} + 2\pi l_N)^2 + \frac{i}{\hbar}\frac{I}{2\epsilon}(\theta_{N-1} - \theta_{N-2})^2\right\} \exp\left\{\frac{i}{\hbar}\frac{m}{2\epsilon}[\mathbf{R}_N(\theta_N + 2\pi l_N) - \mathbf{R}_{N-1}(\theta_{N-1})]^2\right\}$$

$$\times \exp\left\{\frac{i}{\hbar}\frac{m}{2\epsilon}[\mathbf{R}_{N-1}(\theta_{N-1}) - \mathbf{R}_{N-2}(\theta_{N-2})]^2\right\} \exp\left\{-\frac{i\epsilon}{\hbar}\mathcal{V}(\theta_{N-1},\mathbf{R}_{N-1}(\theta_{N-1}))\right\}.$$

Repeat this procedure for the integrals involving θ_n from $n = N-2$ to $n = 1$, and obtain

$$K(\theta,\mathbf{q},t;\theta',\mathbf{q}',0) = \sum_{l=-\infty}^{\infty} \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \left(\prod_{n=1}^{N-1} d\theta_n\right) \int_{-\infty}^{\infty} \left(\prod_{n=1}^{N-1} d\mathbf{R}_{n-1}(\theta_{n-1})\right) \prod_{n=1}^{N} \sqrt{\frac{I}{2\pi i\epsilon\hbar}} \sqrt{\frac{m}{2\pi i\epsilon\hbar}}^{\mathcal{N}}$$

$$\times \exp\left\{\frac{i}{\hbar}\frac{I}{2\epsilon}(\theta_n - \theta_{n-1} + 2\pi l\delta_{n,N})^2\right\}$$

$$\times \exp\left\{\frac{i}{\hbar}\frac{m}{2\epsilon}[\mathbf{R}_n(\theta_n + 2\pi l\delta_{n,N}) - \mathbf{R}_{n-1}(\theta_{n-1})]^2 - \frac{i\epsilon}{\hbar}\mathcal{V}(\theta_{n-1},\mathbf{R}_{n-1}(\theta_{n-1}))\right\}.$$

Define

$$\mathcal{A}_A = \sqrt{\frac{2\pi i\epsilon\hbar}{I}}, \quad \mathcal{A}_B = \sqrt{\frac{2\pi i\epsilon\hbar}{m}}^{\mathcal{N}}, \tag{A5}$$

$$\int \mathcal{D}\theta = \frac{1}{\mathcal{A}_A} \int_{-\infty}^{\infty} \left(\prod_{n=1}^{N-1} \frac{d\theta_n}{\mathcal{A}_A} \right), \quad \int \mathcal{D}\mathbf{R}(\theta) = \frac{1}{\mathcal{A}_B} \int_{-\infty}^{\infty} \left(\prod_{n=1}^{N-1} \frac{d\mathbf{R}_n(\theta_n)}{\mathcal{A}_B} \right), \tag{A6}$$

$$L_n(\theta_n, \mathbf{R}_n(\theta_n)) = \frac{I}{2} \left(\frac{\theta_n - \theta_{n-1} + 2\pi l \delta_{n,N}}{\epsilon} \right)^2 + \frac{m}{2} \left(\frac{\mathbf{R}_n(\theta_n + 2\pi l \delta_{n,N}) - \mathbf{R}_{n-1}(\theta_{n-1})}{\epsilon} \right)^2 - \mathcal{V}(\theta_{n-1}, \mathbf{R}_{n-1}(\theta_{n-1})), \tag{A7}$$

where

$$L[\theta, \mathbf{R}(\theta)] = \lim_{\epsilon \to 0} L_n[\theta_n, \mathbf{R}_n(\theta_n)] = L_0[\theta] + L_B[\theta, \mathbf{R}(\theta)], \tag{A8}$$

$$L_0[\theta] = \frac{I}{2} \dot{\theta}^2, \tag{A9}$$

$$L_B[\theta, \mathbf{R}(\theta)] = \sum_{j=1}^{\mathcal{N}} \frac{m}{2} \dot{R}_j(\theta)^2 - \sum_{j=1}^{\mathcal{N}} \frac{1}{2} m\omega_j^2 \left(R_j(\theta) - \frac{C_j \theta}{m\omega_j^2} \right)^2 \tag{A10}$$

is the Lagrangian of the coupled system. The action $S[\theta, \mathbf{R}(\theta)]$ is given by

$$S[\theta, \mathbf{R}(\theta)] = \int_0^t L[\theta, \mathbf{R}(\theta)] ds = S_0[\theta] + S_B[\theta, \mathbf{R}(\theta)], \tag{A11}$$

$$S_0[\theta] = \int_0^t d\tau L[\theta], \tag{A12}$$

$$S_B[\theta, \mathbf{R}(\theta)] = \int_0^t d\tau L_B[\theta, \mathbf{R}(\theta)]. \tag{A13}$$

Now, suppose that we initially have the total density operator given by

$$\hat{\rho}_{AB}(0) = \hat{\rho}_A(0)\hat{\rho}_B(0). \tag{A14}$$

Then, the reduced density matrix is

$$\rho(\theta, \phi, t) = \int_{-\infty}^{\infty} d\mathbf{R}(\theta) d\mathbf{Q}(\theta) \delta(\mathbf{R}(\theta) - \mathbf{Q}(\theta)) \rho(\theta, \mathbf{p}, \phi, \mathbf{s}, t)$$

$$= \int_0^{2\pi} d\theta' d\phi' J(\theta, \phi, t; \theta', \phi', 0) \rho_A(\theta', \phi', 0), \tag{A15}$$

where

$$J(\theta, \phi, t; \theta', \phi', 0) = \sum_{l=-\infty}^{\infty} \sum_{l'=-\infty}^{\infty} \int_{\theta'}^{\theta+2\pi l} \mathcal{D}\theta \int_{\phi'}^{\phi+2\pi l'} \mathcal{D}\phi^* \exp \frac{i}{\hbar} (S_0[\theta] - S_0[\phi]) \mathcal{F}[\theta, \phi] \tag{A16}$$

is the propagator of the density matrix,

$$\mathcal{F}[\theta, \phi] = \int_{-\infty}^{\infty} d\mathbf{R}(\theta) d\mathbf{Q}(\phi) d\mathbf{R}'(\theta') d\mathbf{Q}'(\phi') \delta(\mathbf{R}(\theta) - \mathbf{Q}(\phi)) \rho_B(\mathbf{p}', \mathbf{s}', 0)$$

$$\times \int_{\mathbf{R}'(\theta')}^{\mathbf{R}(\theta)} \mathcal{D}\mathbf{R}(\theta) \int_{\mathbf{Q}'(\phi')}^{\mathbf{Q}(\phi)} \mathcal{D}\mathbf{Q}(\phi) \exp \frac{i}{\hbar} \{ S_B[\theta, \mathbf{R}(\theta)] - S_B[\phi, \mathbf{Q}(\phi)] \}$$

is the influence functional, and $Q(\phi) = -(s_j - c\phi)/m\omega_j$. Supposed that the density operator $\hat{\rho}_B$ can be written as a canonical ensemble at $t = 0$. That is,

$$\rho_B = \frac{\exp(-\beta \hat{H}_B)}{\mathrm{tr}_B[\exp(-\beta \hat{H}_B)]}. \tag{A17}$$

Introduce the imaginary time $\tau = -i\hbar\beta$; then the density matrix of the environment at $t = 0$ is

$$\rho_B(\mathbf{p}', \mathbf{s}', 0) = \frac{\langle \mathbf{p}' | \exp\left(-\frac{i\tau}{\hbar} \hat{H}_B\right) | \mathbf{s}' \rangle}{\mathrm{tr}_B\left[\exp\left(-\frac{i\tau}{\hbar} \hat{H}_B\right)\right]} = \frac{K_B(\mathbf{p}', \tau; \mathbf{s}', 0)}{\int d\mathbf{q} K_B(\mathbf{q}, \tau; \mathbf{q}, 0)}.$$

2. Momentum representation of the density matrix of a harmonic oscillator

The propagator of a harmonic oscillator with the Lagrangian

$$L = \frac{m}{2}\dot{x}^2 - \frac{1}{2}m\omega x^2 \tag{A18}$$

has a well-known solution

$$K(x,t;x',0) = \int \mathcal{D}x \exp \frac{i}{\hbar}\int_0^t L \, ds \tag{A19}$$

$$= \sqrt{\frac{m\omega}{2\pi i\hbar \sin \omega t}} \exp\left\{\frac{i}{\hbar}\frac{m\omega}{2\sin \omega t}[(x^2 + x'^2)\cos \omega t - 2xx']\right\}. \tag{A20}$$

$K(p,t;p',0)$ is obtained by a Fourier transformation

$$K(p,t:p',0) = \int_{-\infty}^{\infty}\frac{dx dx'}{2\pi\hbar}K(x,t;x',0)\exp\frac{i}{\hbar}(px - p'x'). \tag{A21}$$

Let $A = \frac{m\omega \cos \omega t}{2\sin \omega t}$ and $B = \frac{m\omega}{2\sin \omega t}$; then

$$K(p,t;p',0) = \frac{F(t)}{\sqrt{4B^2 - 4A^2}}\exp\frac{i}{\hbar}\frac{Ap^2 + Ap'^2 - 2Bpp'}{4B^2 - 4A^2},$$

where

$$4B^2 - 4A^2 = \frac{m^2\omega^2(1 - \cos^2 \omega t)}{\sin^2 \omega t} = m^2\omega^2, \tag{A22}$$

so,

$$K(p,t;p',0) = \frac{F(t)}{m\omega}\exp\frac{i}{\hbar}S[p(t)/m\omega]. \tag{A23}$$

Therefore, using $\tau = -i\hbar\beta$, $\beta = 1/k_B T$, $\nu_j = \hbar\omega_j\beta$, and $i \sinh x = \sin ix$ we have

$$\langle \mathbf{p}'|\exp(-i\tau\hat{H}_B/\hbar)|\mathbf{s}'\rangle = \prod_{j=1}^{\mathcal{N}}\frac{1}{m\omega_j}\sqrt{\frac{m\omega_j}{2\pi\hbar \sinh \nu_j}}\exp-\frac{m\omega_j}{2\hbar \sinh \nu_j}\left[\frac{p'^2_j + s'^2_j}{m^2\omega_j^2}\cosh \nu - 2\frac{p'_j s'_j}{m^2\omega_j^2}\right] \tag{A24}$$

and

$$\text{tr}_B[\exp(-i\tau\hat{H}_B/\hbar)] = \int_{-\infty}^{\infty}d\mathbf{q}\,\langle \mathbf{q}|\exp(-i\tau\hat{H}_B/\hbar)|\mathbf{q}\rangle = \prod_{j=1}^{\mathcal{N}}\frac{1}{2\sinh \frac{\nu_j}{2}}.$$

We are assuming that the environment is not coupled to the system at $t = 0$, so $R'_j = p'_j/m\omega_j$ and $Q'_j = s'_j/m\omega_j$. Therefore,

$$\rho_B(\mathbf{p}',\mathbf{s}',0) = \left(\prod_{j=1}^{\mathcal{N}}\frac{1}{m\omega_j}\right)\rho_B(\mathbf{R}',\mathbf{Q}',0), \tag{A25}$$

where

$$\rho_B(\mathbf{R}',\mathbf{Q}',0) = \prod_{j=1}^{\mathcal{N}}2\sinh\frac{\nu_j}{2}\sqrt{\frac{m\omega_j}{2\pi\hbar \sinh \nu_j}}\exp-\frac{m\omega_j}{2\hbar \sinh \nu_j}\left[(R'^2_j + Q'^2_j)\cosh \nu_j - 2R'_j Q'_j\right]. \tag{A26}$$

Let us rewrite this density matrix into a simpler form. Define $x_{0j} = R'_j + Q'_j$ and $x'_{0j} = R'_j - Q'_j$. Then, using

$$(R'^2_j + Q'^2_j)\cosh \nu_j - 2R'_j Q'_j = \frac{1}{2}(\cosh \nu_j - 1)x_{0j}^2 + \frac{1}{2}(\cosh \nu_j + 1)x'^2_{0j} \tag{A27}$$

and the identity

$$\frac{\cosh \nu_j - 1}{\sinh \nu_j} = \frac{\sinh \nu_j}{\cosh \nu_j + 1} = \tanh\frac{\nu_j}{2}, \tag{A28}$$

we obtain

$$\rho_B(\mathbf{R}',\mathbf{Q}',0) = \prod_{j=1}^{\mathcal{N}}\sqrt{\frac{m\omega_j}{\pi\hbar\mu_j}}\exp-\frac{m\omega_j}{4\hbar}\left[\frac{x_{0j}^2}{\mu_j} + x'^2_{0j}\mu_j\right], \tag{A29}$$

$$\mu_j = \coth\frac{\nu_j}{2}. \tag{A30}$$

3. The influence functional $\mathcal{F}[\theta,\phi]$

Now, we calculate the density functional $\mathcal{F}[\theta,\phi]$ explicitly. We will write $\mathbf{R}$ and $\mathbf{Q}$ instead of $\mathbf{R}[\theta]$ and $\mathbf{Q}[\phi]$ for simplicity, but keep in mind that they depend on θ and ϕ, respectively. Then, the influence functional can be written as

$$\mathcal{F}[\theta,\phi] = \int_{-\infty}^{\infty} d\mathbf{x}_t d\mathbf{x}_t' d\mathbf{x}_0 d\mathbf{x}_0' (1/2)^{\mathcal{N}} \delta(\mathbf{x}_t') \rho_B(\mathbf{R}',\mathbf{Q}',0) \int_{\mathbf{R}'}^{\mathbf{R}} \mathcal{D}\mathbf{R} \int_{\mathbf{Q}'}^{\mathbf{Q}} \mathcal{D}\mathbf{Q}^*$$

$$\times \exp\left\{\frac{i}{\hbar} \int_0^t ds \sum_j \left[\frac{m}{2}(\dot{R}_j^2 - \dot{Q}_j^2) - \frac{1}{2}m\omega_j^2(R_j^2 - Q_j^2) + C_j(\theta R_j - \phi Q_j) - \frac{C_j^2}{2m\omega_j^2}(\theta^2 - \phi^2)\right]\right\}. \quad (A31)$$

To evaluate this we introduce the variables

$$\varphi = \theta + \phi, \quad \varphi' = \theta - \phi,$$
$$x_j = R_j + Q_j, \quad x_j' = R_j - Q_j. \quad (A32)$$

Then, the influence functional becomes

$$\mathcal{F}[\theta,\phi] = \int_{-\infty}^{\infty} d\mathbf{x}_t d\mathbf{x}_t' d\mathbf{x}_0 d\mathbf{x}_0' (1/2)^{\mathcal{N}} \delta(\mathbf{x}_t') \rho_B(\mathbf{R}',\mathbf{Q}',0) \int \mathcal{D}\mathbf{x} \mathcal{D}\mathbf{x}'$$

$$\times \exp\left\{\frac{i}{\hbar} \int_0^t ds \sum_j \left[\frac{m}{2}\dot{x}_j\dot{x}_j' - \frac{1}{2}m\omega_j^2 x_j x_j' + \frac{C_j}{2}(\varphi x_j' + \varphi' x_j) - \frac{C_j^2}{2m\omega_j^2}\varphi'\varphi\right]\right\}$$

$$= \int_{-\infty}^{\infty} d\mathbf{x}_t d\mathbf{x}_t' d\mathbf{x}_0 d\mathbf{x}_0' (1/2)^{\mathcal{N}} \delta(\mathbf{x}_t') \rho_B(\mathbf{R}',\mathbf{Q}',0) \exp\left\{\frac{im}{2\hbar} \sum_j (x_{tj}\dot{x}_{tj}' - x_{0j}\dot{x}_{0j}') - \frac{i}{\hbar}\int_0^t ds \sum_j \frac{C_j^2}{2m\omega_j^2}\varphi'\varphi\right\}$$

$$\times \int \mathcal{D}\mathbf{x} \mathcal{D}\mathbf{x}' \exp\left\{\frac{i}{\hbar}\int_0^t ds \sum_j \left[-g(x_j')x_j + \frac{C_j}{2}\varphi x_j'\right]\right\},$$

where

$$g(x_j') = \frac{m}{2}\ddot{x}_j' + \frac{1}{2}m\omega_j^2 x_j' - \frac{C_j}{2}\varphi'. \quad (A33)$$

We note that the classical solution $x_{j,cl}'$ satisfies the Euler-Lagrange equation

$$g(x_{j,cl}') = 0, \quad g(x_j'(0)) = 0, \quad g(x_j'(t)) = 0. \quad (A34)$$

The path integral over $\mathbf{x}$ can be done first. Calling this integral $I_{\mathbf{x},\mathbf{x}'}$ we obtain

$$I_{\mathbf{x},\mathbf{x}'} = \int \mathcal{D}\mathbf{x} \mathcal{D}\mathbf{x}' \exp\left\{\frac{i}{\hbar}\int_0^t ds \sum_j \left[-g(x_j')x_j + \frac{C_j}{2}\varphi x_j'\right]\right\} \quad (A35)$$

$$= \frac{1}{|\mathcal{A}_B|^2} \int_{-\infty}^{\infty} \left(\prod_{n=1}^{N-1} \frac{d\mathbf{x} d\mathbf{x}'}{|\mathcal{A}_B|^2}\right) \prod_{n=1}^{N} \prod_j \exp\frac{i\epsilon}{\hbar}\left[-g(x_{n,j}')x_{n,j} + \frac{C_j}{2}\varphi x_j'\right] \quad (A36)$$

$$= \frac{1}{|\mathcal{A}_B|^2} \int_{-\infty}^{\infty} \left(\prod_{n=1}^{N-1} \frac{d\mathbf{x}'}{|\mathcal{A}_B|^2}\right) \prod_j \left(\prod_{n=1}^{N-1} \delta\left[\frac{\epsilon}{2\pi\hbar}g(x_{n,j}')\right]\right) \prod_{n=1}^{N} \exp\frac{i\epsilon}{\hbar}\frac{C_j}{2}\varphi x_{n,j}'. \quad (A37)$$

$|\mathcal{A}|^{-2}$ is absorbed into the delta functions. The interpretation of the delta function is to insert the classical solution $x_{j,cl}'$ into x_j' which satisfies $g(x_{j,cl}') = 0$. The outcome is

$$I_{\mathbf{x},\mathbf{x}'} = \frac{1}{|\mathcal{A}_B|^2} \exp\left(\frac{i}{\hbar}\int_0^t ds \sum_j \frac{C_j}{2}\varphi x_{cl}'\right). \quad (A38)$$

Therefore, the influence functional is

$$\mathcal{F}[\theta,\phi] = \int_{-\infty}^{\infty} d\mathbf{x}_t d\mathbf{x}_t' d\mathbf{x}_0 d\mathbf{x}_0' (1/2)^{\mathcal{N}} \delta(\mathbf{x}_t') \rho_B(\mathbf{R}',\mathbf{Q}',0) \exp \frac{i}{\hbar} \frac{m}{2} \sum_j (x_{tj} \dot{x}_{tj}' - x_{0j} \dot{x}_{0j}')$$

$$\times \frac{1}{|\mathcal{A}_B|^2} \exp \left(\frac{i}{\hbar} \int_0^t ds \sum_j \left[-\frac{C_j^2}{2m\omega_j^2} \varphi' \varphi + \frac{C_j}{2} \varphi x_{cl}' \right] \right).$$

The $\mathbf{x}_t$ integral gives a delta function of $\dot{\mathbf{x}}_t'$:

$$\mathcal{F}[\theta,\phi] = \int_{-\infty}^{\infty} d\mathbf{x}_t' d\mathbf{x}_0 d\mathbf{x}_0' (1/2)^{\mathcal{N}} \delta(\mathbf{x}_t') \delta(\dot{\mathbf{x}}_t') \rho_B(\mathbf{R}',\mathbf{Q}',0) \exp \left(\frac{i}{\hbar} \int_0^t ds \sum_j \left[-\frac{C_j^2}{2m\omega_j^2} \varphi' \varphi + \frac{C_j}{2} \varphi x_{cl}' \right] + \frac{i}{\hbar} \frac{m}{2} \sum_j x_{0j} \dot{x}_{0j}' \right).$$

Insert Eq. (A29) for the density matrix and do the $\mathbf{x}_0$ integral; then we obtain

$$\mathcal{F}[\theta,\phi] = \int_{-\infty}^{\infty} d\mathbf{x}_t' d\mathbf{x}_0' \delta(\mathbf{x}_t') \delta(\dot{\mathbf{x}}_t') \prod_j \exp \left[\frac{i}{\hbar} \int_0^t ds \left(-\frac{C_j^2}{2m\omega_j^2} \varphi' \varphi + \frac{C_j}{2} \varphi x_{cl}' \right) - \frac{m\mu_j\omega_j}{4\hbar} \left(\frac{\dot{x}_{0j}'^{\,2}}{\omega_j^2} + x_{0j}'^{\,2} \right) \right].$$

The two delta functions $\delta(\mathbf{x}_t')$ and $\delta(\dot{\mathbf{x}}_t')$ can be interpreted as boundary conditions on the classical solution of $g(\mathbf{x}'(s)) = 0$. The result of doing the remaining integrals would be to substitute the classical solution of x_{cl}', $\dot{x}_{0j}'$, and x_{0j}' which satisfy these boundary conditions. The solution to the classical solution of $g(x) = 0$, that is

$$\ddot{x}_j' + \omega_j^2 x_j' - \frac{C_j}{m} \varphi' = 0, \tag{A39}$$

is

$$x_j(\tau) = -\int_{\tau}^t \frac{C_j^2 \varphi'(s)}{m\omega_j} \sin \omega_j(\tau - s) ds + x_{tj}' \cos \omega_j(t - \tau) - \frac{\dot{x}_{tj}'}{\omega_j} \sin \omega_j(t - \tau). \tag{A40}$$

For our boundary condition this solution reduces to

$$x_j'(\tau) = -\int_{\tau}^t \frac{C_j \varphi'(s)}{m\omega_j} \sin \omega_j(\tau - s) ds, \tag{A41}$$

$$\dot{x}_j'(\tau) = -\int_{\tau}^t \frac{C_j \varphi'(s)}{m} \cos \omega_j(\tau - s) ds. \tag{A42}$$

Therefore, the result of integration is

$$\mathcal{F}[\theta,\phi] = \exp \left\{ -\sum_j \frac{i}{\hbar} \frac{C_j^2}{2m\omega_j} \int_0^t \int_{\tau}^t \varphi(\tau) \left[\frac{2}{\omega_j} \delta(\tau - s) - \sin \omega_j(\tau - s) \right] \varphi'(s) d\tau ds \right\}$$

$$\times \exp \left[-\sum_j \frac{\mu_j}{4\hbar} \int_0^t \int_0^t \frac{C_j^2}{m\omega_j} \varphi'(\tau) \cos \omega_j(\tau - s) \varphi'(s) ds d\tau \right].$$

Since s and τ enter into the second integral symmetrically it can be rewritten

$$\int_0^t \int_0^t E(\tau,s) ds d\tau = 2 \int_0^t \int_0^{\tau} E(\tau,s) ds d\tau. \tag{A43}$$

The first integral can be written

$$\int_0^t \int_{\tau}^t E(\tau,s) ds d\tau = \int_0^t \int_0^{\tau} E(s,\tau) ds d\tau. \tag{A44}$$

Then, the influence functional becomes

$$\mathcal{F}[\theta,\phi] = \exp i \Phi[\theta,\phi], \tag{A45}$$

where the influence phase $\Phi[\theta,\phi]$ can be written as

$$i\Phi[\theta,\phi] = -\frac{i}{\hbar}\int_0^t\int_0^\tau d\tau ds\varphi(s)\alpha_I(\tau-s)\varphi'(\tau) - \frac{1}{\hbar}\int_0^t\int_0^\tau dsd\tau\varphi'(\tau)\alpha_R(\tau-s)\varphi'(s), \tag{A46}$$

$$\alpha_R(\tau-s) = \sum_j \frac{C_j^2}{2m\omega_j}\coth\frac{\hbar\omega_j}{2k_BT}\cos\omega_j(\tau-s), \tag{A47}$$

$$\alpha_I(\tau-s) = \sum_j \frac{C_j^2}{2m\omega_j}\left[\frac{2}{\omega_j}\delta(\tau-s) + \sin\omega_j(\tau-s)\right]. \tag{A48}$$

Therefore, we obtain the density matrix propagator $J(\theta,\phi,t;\theta',\phi',0)$,

$$J(\theta_{\rm f},\phi_{\rm f},t;\theta{\rm i},\phi_{\rm i},0) \tag{A49}$$

$$= \int_{\theta_{\rm i}}^{\theta_{\rm f}} D\theta \int_{\phi_{\rm i}}^{\phi_{\rm f}} D\phi^* \exp\left\{\frac{i}{\hbar}S[\varphi^+,\varphi^-] - \Gamma[\varphi^-]\right\}\Bigg|_{\substack{\varphi^+=(\theta+\phi)/2 \\ \varphi^-=\phi-\theta}}, \tag{A50}$$

the classical action $S[\varphi^+,\varphi^-]$ is

$$S[\varphi,\varphi'] = -\int_0^t d\tau I\dot{\varphi}^+(\tau)\dot{\varphi}^-(\tau)$$

$$+ 2\int_0^t d\tau\int_0^s ds\varphi^-(\tau)\alpha_I(\tau-s)\varphi^+(s),$$

and the $\Gamma[\varphi^-]$ is given by

$$\Gamma[\varphi^-] = \frac{1}{2\hbar}\int_0^t d\tau\int_0^t ds\varphi^-(\tau)\alpha_R(\tau-s)\varphi^-(s).$$

The path integral in (A50) can be done exactly and we obtain

$$J(\theta_{\rm f},\phi_{\rm f},t;\theta_{\rm i},\phi_{\rm i},0) = F^2(t)\exp\left(\frac{i}{\hbar}S[\varphi_{\rm cl}^+,\varphi_{\rm cl}^-] - \Gamma[\varphi_{\rm cl}^-]\right). \tag{A51}$$

$\varphi_{\rm cl}^\pm$ are the classical coordinates obtained from the Euler-Lagrange equation

$$I\ddot{\varphi}_{\rm cl}^-(u) + 2\int_u^t d\tau\varphi_{\rm cl}^-(\tau)\alpha_I(\tau-u) = 0, \tag{A52}$$

$$I\ddot{\varphi}_{\rm cl}^+(u) + 2\int_0^u d\tau\varphi_{\rm cl}^+(\tau)\alpha_I(u-\tau) = 0, \tag{A53}$$

whose solution is given in terms of boundary conditions $\varphi_{\rm i}^\pm = \varphi^\pm(0), \varphi_{\rm f}^\pm = \varphi^\pm(t)$:

$$\varphi_{\rm cl}^+(u) = \kappa_i(u;t)\varphi_{\rm i}^+ + \kappa_f(u;t)\varphi_{\rm f}^+, \tag{A54}$$

$$\varphi_{\rm cl}^-(u) = \kappa_i(t-u;t)\varphi_{\rm f}^- + \kappa_f(t-u;t)\varphi_{\rm i}^-, \tag{A55}$$

$$\kappa_i(u;t) = \dot{G}(u) - \frac{\dot{G}(t)}{G(t)}G(u), \quad \kappa_f(u;t) = \frac{G(u)}{G(t)}. \tag{A56}$$

Then, the classical action reduces to

$$S[\varphi_{\rm cl}^+,\varphi_{\rm cl}^-] = -I[\dot{\varphi}_{\rm cl}^+(t)\varphi_{\rm f}^- - \dot{\varphi}_{\rm cl}^-(0)\varphi_{\rm i}^-]. \tag{A57}$$

---

[1] F. Wilczek, Phys. Rev. Lett. **109**, 160401 (2012).

[2] T. Li, Z.-X. Gong, Z.-Q. Yin, H. T. Quan, X. Yin, P. Zhang, L.-M. Duan, and X. Zhang, Phys. Rev. Lett. **109**, 163001 (2012).

[3] G. E. Volovik, JETP Lett. **98**, 491 (2013).

[4] A. J. Leggett, S. Chakravarty, A. T. Dorsey, M. P. A. Fisher, A. Garg, and W. Zwerger, Rev. Mod. Phys. **59**, 1 (1987).

[5] G. Grüner, *Density Waves in Solids* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1994); S. Kagoshima, H. Nagasawa, and T. Sambongi, *One-Dimensional Conductors* (Springer-Verlag, 1988); P. Monceau, Adv. Phys. **61**, 325 (2012).

[6] A. Zettl, G. Grüner, and A. H. Thompson, Phys. Rev. B **26**, 5760 (1982).

[7] S. Tanda, T. Tsuneta, Y. Okajima, K. Inagaki, K. Yamaya, and N. Hatakenaka, Nature (London) **417**, 397 (2002).

[8] T. Matsuura, K. Inagaki, and S. Tanda, Phys. Rev. B **79**, 014304 (2009).

[9] M. Tsubota, K. Inagaki, T. Matsuura, and S. Tanda, Europhys. Lett. **97**, 57011 (2012).

[10] A. Caldeira and A. Leggett, Physica A **121**, 587 (1983).

[11] U. Weiss, *Quantum Dissipative Systems* (World Scientific Publishing Company, Singapore, 1999).

[12] P. Bruno, Phys. Rev. Lett. **111**, 070402 (2013).

[13] H. Watanabe and M. Oshikawa, Phys. Rev. Lett. **114**, 251603 (2015).

[14] C. W. von Keyserlingk and S. L. Sondhi, Phys. Rev. B **93**, 245146 (2016); N. Y. Yao, A. C. Potter, I.-D. Potirniche, and A. Vishwanath, Phys. Rev. Lett. **118**, 030401 (2017); D. V. Else, B. Bauer, and C. Nayak, Phys. Rev. X **7**, 011026 (2017).

[15] S. Choi, J. Choi, R. Landig, G. Kucsko, H. Zhou, J. Isoya, F. Jelezko, S. Onoda, H. Sumiya, V. Khemani, C. von Keyserlingk, N. Y. Yao, E. Demler, and M. D. Lukin, Nature (London) **543**, 221 (2017); J. Zhang, P. W. Hess, A. Kyprianidis, P. Becker, A. Lee, J. Smith, G. Pagano, I.-D. Potirniche, A. C. Potter, A. Vishwanath, N. Y. Yao, and C. Monroe, *ibid.* **543**, 217 (2017).

[16] E. N. Bogachek, I. V. Krive, I. O. Kulik, and A. S. Rozhavsky, Phys. Rev. B **42**, 7614 (1990).

[17] Y. Ohnuki and S. Kitakado, J. Math. Phys. **34**, 2827 (1993).

[18] J. J. Sakurai, *Modern Quantum Mechanics*, rev. ed. (Addison-Wesley Publishing company, Reading, Massachusetts, 1993).

[19] B. Hall, *Quantum Theory for Mathematicians*, Graduate Texts in Mathematics (Springer, New York, 2013).

[20] P. Hänggi, Generalized Langevin equations: A useful tool for the perplexed modeller of nonequilibrium fluctuations? in *Stochastic Dynamics*, Lecture Notes in Physics, Vol. 484 (Springer, Berlin, 1997), pp. 15–22.

[21] H. Grabert, P. Schramm, and G.-L. Ingold, Phys. Rep. **168**, 115 (1988); P. Schramm and H. Grabert, J. Stat. Phys. **49**, 767 (1987).

[22] P. A. Lee, T. M. Rice, and P. W. Anderson, Solid. State Commun. **14**, 703 (1974).

[23] K. Nomura and K. Ichimura, J. Vac. Sci. Technol. A **8**, 504 (1990).

[24] M. Ido, Y. Okajima, and M. Oda, J. Phys. Soc. Jpn. **55**, 2106 (1986).

[25] F. Wilczek, Phys. Rev. Lett. **111**, 250402 (2013).

[26] A. V. Ustinov, T. Doderer, R. P. Huebener, N. F. Pedersen, B. Mayer, and V. A. Oboznov, Phys. Rev. Lett. **69**, 1815 (1992).

[27] H.-K. Li, E. Urban, C. Noel, A. Chuang, Y. Xia, A. Ransford, B. Hemmerling, Y. Wang, T. Li, H. Häffner, and X. Zhang, Phys. Rev. Lett. **118**, 053001 (2017).

[28] V. J. Emery and J. D. Axe, Phys. Rev. Lett. **40**, 1507 (1978).

[29] I. U. Heilmann, J. D. Axe, J. M. Hastings, G. Shirane, A. J. Heeger, and A. G. MacDiarmid, Phys. Rev. B **20**, 751 (1979).

[30] R. V. Pai and R. Pandit, Phys. Rev. B **71**, 104508 (2005).

[31] R. Blinc and A. P. Levanyuk, *Incommensurate Phases in Dielectrics: Parts 1 and 2* (North Holland, Amsterdam, 1986).